Quadratic Singular Perturbation Problems With Nonmonotone Transition Layer Properties

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Abstract: In this article, we consider the quadratic singular perturbation problems with Nonmonotone Transition Layer Properties. Under certain conditions, solutions are shown to exhibit nonmonotone transition layer behavior at turning point *t*=0. The formal approximation of problems is constructed using composite expansions, and then approximation solutions of left and right sides at *t*=0 are joined by joint method which exhibits spike layer behavior and boundary layer behavior respectively. As a result, an approximate solution is formed which exhibits nonmonotone transition layer behavior. In addition, the existence and asymptotic behavior of solutions are proved by the theory of differential inequalities.

Keywords: Singular perturbation; Quadratic problems; Nonmonotone transition layer; The method of composite expansions; Differential inequalities
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1. Introduction

In this article, we address the quadratic singular perturbation problems of the form

$$\varepsilon^2 x'' + f(t)x'^2 + g(t,x) = 0, \quad a < t < b \qquad (1)$$

$$x(a,\varepsilon) = A, \quad x(b,\varepsilon) = B \qquad (2)$$

where $\varepsilon > 0$ is a small parameter, a, b $(a < 0 < b)$ and A, B are constants.

It is well-known that such a singularly perturbed problem has solutions with interesting characteristics like boundary layers and interior layers. Howes [1] has proved the presence of a solution to the problem (1), (2) exhibiting boundary layer behavior under the hypothesis that $f(t)$ is either positive or negative everywhere in $[a,b]$. In his other work [2,3], Howes showed that solutions of (1), (2) can exhibit shock layer behavior in neighborhood of points where f vanishes. Other types of interior layers are also seen, including spike layers and nonmonotone transition layers. Spike layer behavior can be described as non-uniform limiting behavior of a solution in which the solution has an interior maximum or minimum inside the layer. In [4], Feng and Liu studied problem (1), (2) using the method of differential inequalities, and gave sufficient conditions for spike layer behavior. An analysis of nonmonotone transition layers problems can be discovered in DeSanti [5] and Liu [6], who studied the general quasilinear problem.

The present paper is concerned with nonmonotone transition layer phenomena of problems (1), (2), which occur when f has a certain type of turning points at (a,b), say $t=0$. We say that a solution $x=x(t,\varepsilon)$ to problem(1), (2) exhibits nonmonotone transition layer behavior at $t=0$ if

$$\lim_{\varepsilon\to 0}x(t,\varepsilon)=\begin{cases}u_L(t), & a\le t<0,\\ s, & t=0,\\ u_R(t), & 0<t\le b,\end{cases} \tag{3}$$

where $u_L(0)\neq u_R(0)$, and $s>\max\{u_L(0),u_R(0)\}$ or $s<\min\{u_L(0),u_R(0)\}$.

We have constructed a formal approximation using the method of composite expansion [7]. It is then shown, using the theory of differential inequalities [8], that problem (1), (2) has a solution with the desired properties.

2. The Formal Approximation

For simplicity, we have considered only zero-order approximations of solutions, although higher order approximations can be constructed and verified using the same methods. Assume

$[H_1]$ there exist functions $u_L(t)$ and $u_R(t)$ of C^2 on $[a,b]$ satisfying the reduced problems

$$f(t)u'^2+g(t,u)=0,\quad u(a)=A \tag{4}$$

and

$$f(t)u'^2+g(t,u)=0,\quad u(b)=B \tag{5}$$

respectively，so that $u_L(0)\neq u_R(0)$;

$[H_2]$ $f(t)\in C^n[a,b]\,(n\ge 3)$ satisfying $f(0)=f'(0)=\Lambda=f^{(n-1)}(0)=0$ and $f^{(n)}(0)\neq 0$;

$[H_3]$ $g(t,x)\in C^1([a,b]\times R)$, and there occurs a number $l>0$ such that $g_x(0,x)\le -l$.

First, we search an outer solution in the form

$$U(t,\varepsilon)=\sum_{j=0}^{\infty}u_j(t)\varepsilon^j \tag{6}$$

Substituting (6) into (1) and (2), and equating coefficients of ε^0, we see that $u_0=u_L(t)$ and $u_0=u_R(t)$ solve less problems (4) and (5) respectively. Hence, we can take $u(t)=u_L(t)\ (a\le t\le b)$ or $u(t)=u_R(t)\ (a\le t\le b)$ as a zero-order approximation of the outer solution.

Since $u_L(0)\neq u_R(0)$， to be able to construct correction terms near $t=0$ in the form

$$V(\xi,\varepsilon)=\sum_{j=0}^{\infty}v_j(\xi)\varepsilon^j \tag{7}$$

with $\xi=\frac{t}{\varepsilon}$. Substituting (6) and (7) into (1), we get

$$\ddot V+\frac{2}{\varepsilon}f(\xi\varepsilon)U'\dot V+\frac{1}{\varepsilon^2}f(\xi\varepsilon)\dot V^2+[g(\xi\varepsilon,U+V)-g(\xi\varepsilon,U)]=0, \tag{8}$$

where $\dot V=\frac{dV}{d\xi}$ and $\ddot V=\frac{d^2V}{dx^2}$. From hypothesis $[H_2]$, it follows that

$$f(\xi\varepsilon)=\frac{f^{(n)}(\theta\xi\varepsilon)}{n!}(\xi\varepsilon)^n\,(n\ge 3,\ 0<\theta<1) \tag{9}$$

Equating the coefficient of ε^0 in (8) yields

$$\ddot v_0+\tilde g(v_0)=0, \tag{10}$$

where $\tilde g(v_0)=g(0,u_0(0)+v_0)-g(0,u_0(0))$, $u_0(0)=u_L(0)$ or $u_R(0)$.

Without loss of generality, we choose $u_0(0)=u_L(0)$ in (10)，and write the correction term $v_0=v_L(\xi)$ corresponding to $u_L(t)$ .Since a spike is assumed to be at $t=0$, $v_L(\xi)$ must satisfy the following conditions

$$v_L(0)=c\neq 0,\quad \dot v_L(0)=0, \tag{11}$$

and

$$v_L(-\infty)=\dot v_L(-\infty)=0$$

. (12)

It follows from (10) and (12) that

$$\dot v_L^2 = -2\int_0^{v_L}[g(0,u_L(0)+z)-g(0,u_L(0))]dz \quad (13)$$

And from the second formula of (11), $\dot v_L(0)=0$, yields

$$\int_0^{v_L(0)}[g(0,u_L(0)+z)-g(0,u_L(0))]dz = 0 \quad (14)$$

Consequently,

$$\int_0^{v_L}[g(0,u_L(0)+z)-g(0,u_L(0))]dz < 0 \quad (15)$$

by using (15), we get

$$G(v_L) = -\int_0^{v_L}[g(0,u_L(0)+z)-g(0,u_L(0))]dz \quad (16)$$

From (13), we see that $v_L(\xi)$ could be expressed implicitly

$$|\xi| = \begin{cases} \int_{v_L}^{v_L(0)} \dfrac{dz}{\sqrt{2G(z)}}, & 0 < v_L < v_L(0) \\ \int_{v_L(0)}^{v_L} \dfrac{dz}{\sqrt{2G(z)}}, & v_L(0) < v_L < 0 \end{cases} \quad (17)$$

In order to illuminate $v(0) = c \neq 0$ (without loss of generality let $c>0$), the following known result in Clement and Sweers [9] need to be used.

Lemma 1. Let $h(w)$ be a continuously differential function satisfying that

$[A_1]$ there are numbers $w_2 > w_1 > 0$ so that $h(w_1) = h(w_2) = 0$, $h(w) > 0$ for $w_1 < w < w_2$, h changes sign at w_1, and $h(0) \geq 0$;

$[A_2]$ $\int_q^{w_2} h(w)dw > 0, \quad 0 \leq q \leq w_2$,

Then, there is a $r>0$ so that the following problem has a C^2, radially symmetric solution $w(r)$:

$$\begin{cases} \rho\Delta w + h(w) = 0 \\ w(0) \in (w_1, w_2) \\ w(1) = -1 \\ w'(r) < 0, \quad r > 0 \end{cases} \quad (18)$$

The function $\tilde g(v_L) = g(0,u_L(0)+v_L) - g(0,u_L(0))$ satisfies conditions $[A_1]$ and $[A_2]$ in **Lemma 1** has been by Feng and Liu [4]. Define

$$\varphi(\xi) = w(r) \quad (19)$$

where $r = \sqrt{\rho}\,\xi$, w and ρ are given by **Lemma 1**. Then

$$\rho\Delta w = \rho\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = \ddot\varphi + \frac{1}{\xi}\dot\varphi \quad (20)$$

Hence, by **Lemma 1** and (20), we get

$$\ddot\varphi + \frac{1}{\xi}\dot\varphi + \tilde g(\varphi) = 0 \quad (21)$$

Since $\dot\varphi < 0$ for $x>0$, (21) can be written as

$$\ddot f + \tilde g(f) = -\frac{1}{x}\dot f > 0 \quad (22)$$

Using the symmetry of φ to $\xi = 0$, $\varphi(\xi)$ is a lower solution of (10). Also, obviously $\psi(\xi) \equiv w_2$ is an upper solution of (10). Thus, we have

$$\varphi(\xi) \leq v_L(\xi) \leq \psi(\xi) \quad (23)$$

In particular,

$$\varphi(0) \leq v_L(0) \leq w_2 \quad (24)$$

This means that $v_L(\xi)$ has a spike at $x=0$.

Again, Feng and Liu [4] has provided that given any $0<d<k$,

$$v_L(\xi) = O(\exp(-(k-\delta)|\xi|))\ (\xi \to \infty) \quad (25)$$

where $k = \sqrt{-\tilde g_x(0)} = \sqrt{-g_x(0,u_L(0))} > 0$. In other words, $v_L(\xi)$ is exponentially small term as $\xi \to \infty$.

Consequently, here we can obtain a zero-order approximation of equation (1) satisfying the reduced problems (4)

$$x_0(t,\varepsilon)=u_L(t)+v_L(\frac{t}{\varepsilon}) \tag{26}$$

which has a spike at $t=0$.

Next, we turn to construct the boundary layer correction terms using reduced solution $u_0=u_R(t)$. Taking $u_0(0)=u_R(0)$ in (10), we write the correction term $v_0=v_R(\xi)$ corresponding to $u_R(t)$. Since a spike is assumed to be at $t=0$, $v_R(\xi)$ must satisfy the following conditions

$$v_R(0)=c+u_L(0)-u_R(0),\quad \dot{v}_R(0^+)=0 \tag{27}$$

$$v_R(+\infty)=\dot{v}_R(+\infty)=0 \tag{28}$$

where $c>|u_L(0)-u_R(0)|$ or $c<-|u_L(0)-u_R(0)|$. We consider only the case that $u_L(0)>u_R(0)$ and $c>u_L(0)-u_R(0)$, since the other case is similar.

As discussed before, taking $v_0(\xi)=v_R(\xi)$ in (10), and using conditions (25), (27), we see that the solution $v_R(x)$ on $[0,+\infty)$ could be expressed implicitly as

$$\xi=\int_{v_R(0)}^{v_R}\frac{dz}{\sqrt{2H(z)}} \tag{29}$$

in which

$$H(z)=-\int_0^z[g(0,u_R(0)+z)-g(0,u_R(0))]dz \tag{30}$$

Similarly, it turns out that given any $0<\delta_1<k_1$,

$$v_R(\xi)=O(\exp(-(k_1-\delta_1)\xi))\ (\xi\to+\infty) \tag{31}$$

Where $k_1=\sqrt{-g_x(0,u_R(0))}>0$. In other words, $v_R(\xi)$ is exponentially small term As $\xi\to+\infty$.

Finally, we join the approximation solutions of left and right sides at $t=0$ by joint method which exhibits spike layer behavior and boundary layer behavior respectively. This leads to constructing a zero-order approximation of problem (1), (2) in the form

$$x_0(t,\varepsilon)=\begin{cases}u_L(t)+v_L(\frac{t}{\sqrt{\varepsilon}}), & a\le t\le 0,\\ u_R(t)+v_R(\frac{t}{\sqrt{\varepsilon}}), & 0<t\le b,\end{cases} \tag{32}$$

where $u_L(t)$ and $u_R(t)$ solve the reduced problems (4) and (5), v_L and v_R are given by (17) and (29) satisfying joint condition

$$v_R(0)=c+u_L(0)-u_R(0). \tag{33}$$

3. The existence and asymptotic behavior of solutions

Using the theory of differential inequalities, a solution to the problem (1), (2) was shown, which exhibits nonmonotone transition layer behavior at t = 0. An important consequence of this method is that in the course of proving the existence of solutions, we obtain simultaneously an evaluation of this solution in terms of the solutions of appropriate inequalities.

Theorem 1. Assume $[H_1]-[H_3]$, under the condition (33). Then, for ε sufficiently small, say $0<e<e_0$, problem (1), (2) has a solution $x(t,\varepsilon)$ with

$$x(t,\varepsilon)=x_0(t,\varepsilon)+O(\varepsilon) \tag{34}$$

as $\varepsilon\to 0$, uniformly on $[a,b]$. More precisely, $x(t,\varepsilon)\to u_L(t)$ for x in $[a,0)$, $x(t,\varepsilon)\to u_R(t)$ for x in $(0,b]$ and $x(0,\varepsilon)\to u_L(0)+v_L(0)$ as $\varepsilon\to 0$, where $v_L(0)>0$ if $u_L(0)>u_R(0)$. It is to say $x(t,\varepsilon)$ shows nonmonotone transition layer behavior at $t=0$.

Proof. We consider only the case $u_L(0)>u_R(0)$ and $c>u_L(0)-u_R(0)$.

First, we claim that there exists a solution on $[a,b]$ of the boundary value problem

$$\varepsilon^2x''+f(t)x'^2+g(t,x)=0,\quad a<t<b, \tag{35}$$

$$x(a,\varepsilon)=A,\quad x(b,\varepsilon)=u_L(b)$$

(36)

Defining on the interval $[a,b]$,

$$\alpha(t,\varepsilon)=u_L(t)+v_L(\frac{t}{\varepsilon})-r\varepsilon$$

(37)

$$\beta(t,\varepsilon)=u_L(t)+v_L(\frac{t}{\varepsilon})+r\varepsilon$$

(38)

where $r>0$ is a constant to be determined below. From the construction of formal approximation, by using (35) and (37), we have

$$\varepsilon^2\alpha''+f(t)\alpha'^2+g(t,\alpha)$$

$$=\ddot{v}_L+\tilde{g}(v_L)-g_x(0,u_L(0))r\varepsilon+O(\varepsilon)$$

(39)

It follows from (10) and $[H_3]$ that

$$\varepsilon^2\alpha''+f(t)\alpha'^2+g(t,\alpha)\ge(lr-K)\varepsilon\ge 0\ ,\quad a\le t\le b\ ,$$

(40)

if r is chosen so that $r\ge\frac{K}{l}$, and $K>0$ satisfies $|O(\varepsilon)|\le K\varepsilon$.

Use the similar process of (40), we have

$$\varepsilon^2\beta''+f(t)\beta'^2+g(t,\beta)\le(K-lr)\varepsilon\le 0$$

(41)

Moreover, clearly, $\alpha(t,\varepsilon)\le\beta(t,\varepsilon)$ for $a\le t\le b$, and by (25), $v_L(\frac{a}{\varepsilon})$ and $v_L(\frac{b}{\varepsilon})$ are exponentially small terms as $\varepsilon\to 0$, thus, for ε sufficiently small, yields

$$\alpha(a,\varepsilon)\le A\le\beta(a,\varepsilon)$$

(42)

$$\alpha(b,\varepsilon)\le B\le\beta(b,\varepsilon)$$

(43)

According to the theory of differential inequalities [8], we concluded that there is a solution $x=\mu(t,\varepsilon)$ to problem (35), (36) so that

$$\alpha(t,\varepsilon)\le\mu(t,\varepsilon)\le\beta(t,\varepsilon),\quad a\le t\le b$$

(44)

Thus, it can be seen that

$$x(t,\varepsilon)=u(t)+v(\frac{t}{\varepsilon})+O(\varepsilon)$$

(45)

as $\varepsilon\to 0$, uniformly on $[a,b]$.

Specifically, by (44), we get

$$\alpha(0,\varepsilon)\le\mu(0,\varepsilon)\le\beta(0,\varepsilon)$$

(46)

Next, it can be shown that there exists a solution on $[0,b]$ of the boundary value problem

$$\varepsilon^2x''+f(t)x'^2+g(t,x)=0,\quad 0<t<b\ ,$$

(47)

$$x(0,\varepsilon)=\mu(0,\varepsilon),\quad x(b,\varepsilon)=B\ .$$

(48)

Defining on the interval $[0,b]$,

$$\alpha_1(t,\varepsilon)=u_R(t)+v_R(\frac{t}{\varepsilon})-r_1\varepsilon\ ,$$

(49)

$$\beta_1(t,\varepsilon)=u_R(t)+v_R(\frac{t}{\varepsilon})+r_1\varepsilon\ ,$$

(50)

where $r_1>0$ is a constant to be determined below. From the construction of formal approximation, by using (47) and (49), we have

$$\varepsilon^2\alpha_1''+f(t)\alpha_1'^2+g(t,\alpha_1)$$

$$=\ddot{v}_R+\tilde{g}(v_R)-g_x(0,u_R(0))r_1\varepsilon+O(\varepsilon)$$

(51)

It follows from (10) and $[H_3]$ that

$$\varepsilon^2\alpha_1''+f(t)\alpha_1'^2+g(t,\alpha_1)\ge(lr_1-K)\varepsilon\ge 0\ ,\quad a\le t\le b\ ,$$

(52)

if r_1 is chosen so that $r_1\ge\frac{K_1}{l}$, and $K_1>0$ satisfies $|O(\varepsilon)|\le K_1\varepsilon$.

Use the similar process of (52), we have

$$\varepsilon\beta_1''+f(t)\beta_1'^2+g(t,\beta_1)\le(K-lr_1)\varepsilon\le 0\ .$$

(53)

Moreover, clearly, $\alpha_1(t,\varepsilon) \le \beta_1(t,\varepsilon)$ for $a \le t \le b$, and by (31), $v_R(\frac{b}{\varepsilon})$ is exponentially small term as $\varepsilon \to 0$, thus, for ε sufficiently small, yields

$$\alpha_1(b,\varepsilon) \le B \le \beta_1(b,\varepsilon). \tag{54}$$

Again, note that $\alpha_1(0,0)=\alpha(0,0)$, $\beta_1(0,0)=\beta(0,0)$, according to (46), satisfies

$$\alpha_1(0,\varepsilon) \le \mu(0,\varepsilon) \le \beta_1(0,\varepsilon) \tag{55}$$

for ε sufficiently small. Based on the theory of differential inequalities [8], we arrived at a conclusion that there is a solution $x = \upsilon(t,\varepsilon)$ to problem (47), (48) so that

$$\alpha_1(t,\varepsilon) \le \upsilon(t,\varepsilon) \le \beta_1(t,\varepsilon), \quad 0 \le t \le b \tag{56}$$

Let us finally define

$$x(t,\varepsilon) = \begin{cases} \mu(t,\varepsilon), & a \le t \le 0, \\ \upsilon(t,\varepsilon), & 0 < t \le b, \end{cases} \tag{57}$$

Putting everything together, we see that $x(t,\varepsilon)$ is a solution to problem (1), (2), with

$$x(t,\varepsilon) = x_0(t,\varepsilon) + O(\varepsilon) \tag{58}$$

as $\varepsilon \to 0$, uniformly on $[a,b]$. More precisely, $x(t,\varepsilon) \to u_L(t)$ for x in $[a,0)$, $x(t,\varepsilon) \to u_R(t)$ for x in $(0,b]$ and $x(0,\varepsilon) \to u_L(0) + v_L(0)$ as $\varepsilon \to 0$,

Since

$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = \begin{cases} u_L(t), & a \le t < 0, \\ s, & t = 0, \\ u_R(t), & 0 < t \le b, \end{cases} \tag{59}$$

where $s = u_L(0) + v_L(0)$. When $u_L(0) > u_R(0)$, we have $v_L(0) > 0$ and the result is $s > \max\{u_L(0), u_R(0)\}$. That is to say, $x(t,e)$ exhibits nonmonotone transition layer behavior at $t = 0$, and the proof is complete.

4. Examples

Consider the boundary value problem

$$\varepsilon^2 x'' + t^3 x'^2 + t^2 x(x-\pi) - x\sin x = 0, \quad -1 < t < 1, \tag{60}$$

$$x(-1,\varepsilon) = 0, \quad x(1,\varepsilon) = \pi. \tag{61}$$

Here $f(t) = t^3$, $g(t,x) = t^2 x(x-\pi) - x\sin x$. The reduced problems are

$$t^3 u'^2 + t^2 u(u-\pi) - u\sin u = 0, \quad u(-1) = 0, \tag{62}$$

$$t^3 u'^2 + t^2 u(u-\pi) - u\sin u = 0, \quad u(1) = \pi. \tag{63}$$

These problems have solutions $u_L(t) \equiv 0$ and $u_R(t) \equiv \pi$ respectively，so that $u_L(0) \neq u_R(0)$. It is thus clear that conditions of assume $[H_1]-[H_3]$ are satisfied.

Now, $\tilde{g}(v_L) = -v_L \sin v_L$ satisfies $\tilde{g}(\pi) = \tilde{g}(2\pi) = 0$, $\tilde{g}(v_L) > 0$ for $\pi < v < 2\pi$, and

$$\int_{v_L}^{2\pi} \tilde{g}(z)dz = \int_{v_L}^{2\pi} (-z\sin z)dz = 2\pi - v_L\cos v_L + \sin v_L > 0 \tag{64}$$

for $0 \le v_L < 2\pi$. The function $\tilde{g}(v_L)$ satisfies conditions $[A_1]$ and $[A_2]$ in **Lemma 1**. Define

$$F(v_L) = \int_0^{v_L} \tilde{g}(z)dz = v_L\cos v_L - \sin v_L \tag{65}$$

then $F(\pi) = -\pi < 0$, $F(\frac{3\pi}{2}) = 1 > 0$. Thus, by the intermediate value theorem there is $\pi < c < \frac{3\pi}{2}$ so that $F(c) = 0$. Consequently, all conditions of **Theorem 1** are satisfied. We deduced that problem (60), (61) has a solution $x(t,\varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = \begin{cases} 0, & a \le t < 0, \\ c, & t = 0, \\ \pi, & 0 < t \le b, \end{cases} \tag{66}$$

Since $\pi < c < \frac{3\pi}{2}$, it means that $x(t,\varepsilon)$ exhibits nonmonotone transition layer behavior at $t = 0$.

Today, the issue of nano is one of the serious issues that is used in many fields and industries

[10-16]

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