Angular momentum transport at a planet in the gaseous protoplanetary disk: a new version of an old model

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Received: 06/ 2023, Published: 07/2023

Abstract

The evolution of the disk is influenced by both internal viscous torques and external torques brought on by one or more embedded planets. As planets form and grow within gaseous protoplanetary disks, the mutual gravitational interaction between the disk and planet leads to the exchange of angular momentum and migration of the planet. The functional $\Lambda(R)$depends on the tidal dissipation distribution in the disk which is concentrated in a vicinity ofthe protoplanets orbit. The aim of this work is to solve the equation for the evolution of the surface density of the disk according to the behavior of the angular momentum.

Keywords: planet, angular momentum, protoplanetary disk, viscous torques.

Tob Regul Sci. ™ 2023;9(1): 3320-3330
DOI: doi.org/10.18001/TRS.9.1.233

1. INTRODUCTION

Understanding the evolution of protoplanetary disks and the intricate dynamics within them is essential for unraveling the mysteries of planet formation and migration. These disks, composed of gas and dust surrounding young stars, serve as the birthplace for planets. The interaction between planets and the surrounding disk plays a crucial role in shaping their orbits and determining their ultimate destinations within the planetary system.

Recent reviews have extensively covered the orbital evolution of planets resulting from their interactions with the ambient disk [1]. The gravitational forces at play between a planet and the disk can have profound effects on the planet's orbit. Not only can they stimulate or inhibit orbital deviations, but they can also alter the size of the planet's orbit, leading to migration either closer to or farther away from its host star [2]. Remarkably, these changes can occur rapidly, on a timescale significantly shorter than the time required for planets to fully form.

When a planet traverses its orbit within the disk, it creates disturbances in the surrounding gas and planetary material, giving rise to waves of spiral density. These density waves engender a delicate interplay between the forces exerted by the spirals within and outside the planet's orbit. Consequently, an imbalance may arise, resulting in the planet gaining or losing angular momentum [3]. Such changes in angular momentum have a direct impact on the planet's motion within the disk. If there is a loss of motion, the planet tends to migrate inward, while a gain in motion leads to outward migration [4-8].

The formation of gaps in the disk is another intriguing consequence of the interaction between planets and the circumstellar material. As an accreting planet reaches sufficient mass, it creates a gap in its disk. The massive planet and the circumstellar disk undergo tidal interactions, leading to the transfer of angular momentum between them [1, 3, 4, 9, 10]. The motion of the planet within the disk excites density waves both inside and outside its orbit, resulting in the clearing of material and gap formation, ultimately terminating the accretion process.

The evolution of protoplanetary disks is driven by the viscous transfer of angular momentum by the central star, and the evolution of gas surface density within the disk is described by a diffusive-type equation [1]. In the context of disk migration, a giant planet gradually spirals toward its host star due to tidal interactions within the protoplanetary disk, ultimately reaching the inner edge of the disk [13].

This work aims to solve the equation governing the evolution of the disk's surface density based on the behavior of angular momentum in different regions: $R < a$ and $R > a$. Unlike previous research, this study avoids making approximations on the angular momentum $\Lambda(r)$, thereby distinguishing itself [12]. In Section 2, we present the evolution equations of the protoplanetary disk, along with the angular momentum calculation required for determining the disk's surface density. Section 3 introduces the solution to the evolution equation based on the behavior of angular momentum in different regions. Finally, the last section presents the significant findings of this study, culminating in a comprehensive conclusion.

2. SPECIFIC ANGULAR MOMENTUM AND PLANET MIGRATION MODEL

We consider the total torque as the combination of three distinct resonance contributions: Firstly, the inner Lindblad resonances generate a partial torque that induces outward migration. Secondly, the outer Lindblad resonances contribute to inward migration. Lastly, the corotation resonance also contributes to the total torque. However, it is challenging to predict the migration direction accurately in analytical calculations due to the requirement of precise torque calculations. Additionally, actual protoplanetary disks can exhibit turbulence, resulting in fluctuating torques caused by turbulent density variations.

In this study, we present models that depict the migration of giant planets within evolving protoplanetary disks, where the disk's angular momentum transport is influenced by the central star's viscous effects. The migration of planets occurs in the Type II migration regime, primarily driven by tidal interactions with the disk. Moreover, the disk is subject to tidal torques exerted by the planets [1]. The coupled evolution of the protoplanetary disk and planet is described by an equation that accounts for their mutual influence and progression:

$$\frac{\partial\Sigma}{\partial t}=\frac{1}{R}\left[3R^{1/2}\frac{\partial}{\partial R}\left(\eta\Sigma R^{1/2}\right)-\frac{2\Lambda\Sigma R^{2/3}}{(GM)^{1/2}}\right] \quad (1)$$

Where $\sum(R,t)$ is the disk surface density, t is time, R is cylindrical radius, η is the kinematic viscosity, and M is the stellar mass. The first term on the right hand side describes ordinary viscous evolution of the disk [15] and the second term describes how the disk responds to the planetary torque. Here Λ is the injection rate of angular momentum per unit mass into the disk. Following [16] for a planet of mass $M_p = qM$in circular orbit at radius a, the torque distribution has the form

$$\Lambda(R)=\begin{cases}\frac{-q^2GM}{2R}\left(\frac{R}{\Delta p}\right)^4 & \text{if} \quad R<a\\ \frac{q^2GM}{2R}\left(\frac{R}{\Delta p}\right)^4 & \text{if} \quad R>a\end{cases} \quad (2)$$

Where (Δp) is equal to greater of H and $|R-a|$ whereas H is the scale-height of the disk.

3. SURFACE DENSITY EVOLUTION EQUATION

In our model, we will solve the equation of the evolution equation (2) and find values of torque without numerical calculation solutions, in the special case where the viscosity is a radial power law,$\eta\sim R^n$ with $n<2$ and by assuming a separable ansatz of the form$\sum(R,t)=\varphi(R)\exp(-\lambda t)$, where λ is real number and $\varphi(R)$ is an arbitrary function of R. This work has led to indications of how the density wave propagation, the equation (1) can be rewritten as the differential equation:

$$0=R^2\varphi''(R)+\left(2n+\frac{3}{2}-\frac{2\Lambda R^{\frac{3}{2}}}{\sqrt{GM}3\eta}\right)R\varphi'(R)+\left(n^2+\frac{n}{2}-\frac{2\Lambda' R^{\frac{5}{2}}}{\sqrt{GM}3\eta}-\frac{3}{\sqrt{GM}}\frac{\Lambda R^{\frac{3}{2}}}{3\eta}+\frac{\lambda}{3\eta}R^2\right)\varphi(R) \quad (3)$$

By making the change of variable R to ax, $\varphi(R)$changes to $u(x)$,$\eta(R)$ to $v(x)$,we find the following equation that governs$u(x)$

$$0=\frac{u''(x)}{u(x)}+\left(\frac{3}{2x}+2\frac{v'(x)}{v(x)}-\frac{2\sqrt{x}}{3v(x)}l(x)\right)\frac{u'(x)}{u(x)}+\frac{3}{2x}\frac{v'(x)}{v(x)}+\frac{v''(x)}{v(x)}-\frac{l(x)}{\sqrt{x}v(x)}-\frac{2\sqrt{x}}{3v(x)}l'(x)+\frac{\lambda}{3v(x)} \quad (4)$$

Where:

$$l(x) = \frac{\Lambda a^{\frac{3}{2}}}{\sqrt{GM}} ; \qquad l'(x) = \frac{a^{\frac{3}{2}}}{\sqrt{GM}} \frac{d\Lambda}{dx}$$

We will solve the equation (4) in each region ($x < 1$ and $x > 1$) by giving appropriate parameters in each region. This is the subject of the following two subsections.

3.1 IN THE REGIONI **R < a,x < 1**

The equation (4) has a generic form as:

$$u''(x)+f_1(x)u'(x) + f_0(x)u(x) = 0 \tag{5}$$

Where $f_0(x)$ and $f_1(x)$ are respectively given by:

$$f_0 = \frac{3}{2x}\frac{v\prime(x)}{v(x)} + \frac{v\prime\prime(x)}{v(x)} - \frac{l(x)}{\sqrt{x}v(x)} - \frac{2\sqrt{x}}{3v(x)} l'(x) + \frac{\lambda}{3v(x)} \tag{6}$$

$$f_1 = \frac{3}{2x} + 2\frac{v\prime(x)}{v(x)} - \frac{2\sqrt{x}}{3v(x)} l(x)$$

(7)

To solve the equation (4) put the change:

$$u(x) = \frac{w(x)}{q(x)}$$

(8)

After simplifications, it is traight forward and easy to get

$$\frac{w''(x)}{w(x)} + \left(f_1(x) - \frac{2q'(x)}{q(x)}\right)\frac{w'(x)}{w(x)} + \left[f_0(x) - \frac{q''(x)}{q(x)} + 2\left(\frac{q'(x)}{q(x)}\right)^2 - f_1(x)\frac{q'(x)}{q(x)}\right]=0 \tag{9}$$

By using

$$f_1(x) - \frac{2q'(x)}{q(x)} = \frac{g}{x}$$

(10)

where g is a constant to be determined later. When we replace f_1 in the last equation, we can find $q(x)$, that is to say

$$q(x) = exp\left(\frac{1}{2}\int^{x} f_1(\theta)d\theta - \frac{g}{2}\ln(x)\right)$$

(11)

$$q'(x) = \left(\frac{f_1(x)}{2} - \frac{g}{2x}\right)q(x)$$

(12)

$$q''(x) = \left[\left(\frac{f'_1(x)}{2} + \frac{g}{2x^2}\right) + \left(\frac{f_1(x)}{2} - \frac{g}{2x}\right)^2\right] q(x)$$

(13)

Then the equation (8) transforms to

$$w''(x) + \frac{g}{x} w'(x) + \left[f_0(x) - \frac{q''(x)}{q(x)} + 2\left(\frac{q'(x)}{q(x)}\right)^2 - f_1(x)\frac{q'(x)}{q(x)}\right] w(x) = 0$$

(14)

$$w''(x) + \frac{g}{x} w'(x) + G(x) w(x) = 0$$

(15)

Where g is a free transformation parameter that we will use for our convenience later and

$$G(x) = f_0(x) - \frac{q''(x)}{q(x)} 2\left(\frac{q'(x)}{q(x)}\right)^2 - f_1(x)\frac{q'(x)}{q(x)}$$

(16)

Or equivalently, after using the formula (9) , (12) we find

$$G(x) = f_0(x) - \frac{f'_1(x)}{2} - \frac{f_1^2(x)}{4} + \frac{g(g-2)}{4x^2}$$

(17)

substitute the expressions of$f_0(x)$ and $f_1(x)$ and $f'_1(x)$given above, we find

$$G(x) = \frac{3}{16x^2} + \frac{\lambda}{3v(x)} - \frac{l(x)}{3\sqrt{x}v(x)} - \frac{\sqrt{x}}{3v(x)}\frac{d}{dx} l(x) - \frac{x}{9v^2(x)} l^2(x) \qquad + \frac{\sqrt{x}}{3}\frac{v'(x)}{v^2(x)} l(x) + \frac{g(g-2)}{4x^2}$$

(18)

Replace now the viscosity $v(x)$ by Sx^{β} then:

$$G(x) = \frac{3}{16x^2} + \frac{\lambda}{3Sx^{\beta}} + \frac{g(g-2)}{4x^2} - \frac{1}{3Sx^{\beta-\frac{1}{2}}}\frac{d}{dx} l(x) \qquad - \frac{1}{9S^2x^{2\beta-1}} l^2(x) + \frac{(\beta-1)}{3Sx^{\beta+1/2}} l(x)$$

(19)

Now, to solve the above differential equation (14), we discard the terms containing the tore $l(x)$ in (19), that is to say:

$$0 = \frac{d}{dx} l(x) + \frac{l^2(x)}{3Sx^{\beta-\frac{1}{2}}} - \frac{(\beta-1)}{x} l(x)$$

(20)

This constraint has a twist advantage: the first is to have a freely expression of the tore $l(x)$ that responds to the physical situation in our consideration. The second advantage is a purely mathematical issue that allows as to have analytically the solution of the differential equation (9) that governs$u(x)$ via (8). Indeed, we have from the constraint (20).

$$\mathrm{l(x)} = \frac{3\mathrm{Sx}^{\beta-1}}{3\mathrm{CS}+2\sqrt{\mathrm{x}}}$$

(21)

and

$$\mathrm{G(x)} = \frac{3}{16\mathrm{x}^2} + \frac{\lambda}{3\mathrm{Sx}^\beta} + \frac{\mathrm{g(g-1)}}{4\mathrm{x}^2}$$

(22)

Leading to get a simplified differential equation

$$\mathrm{w''(x)} + \frac{\mathrm{g}}{\mathrm{x}}\mathrm{w'(x)} + \left(\frac{3}{16\mathrm{x}^2} + \frac{\lambda}{3\mathrm{Sx}^\beta} + \frac{\mathrm{g(g-1)}}{4\mathrm{x}^2}\right)\mathrm{w(x)} = 0$$

(23)

For lightrning we put

$$\frac{\lambda}{3S} = \lambda_s$$

(24)

$$\frac{4\mathrm{g(g-2)}+3}{16} = \Gamma$$

(25)

We get

$$w''(x) + \frac{g}{x}w'(x) + \left(\frac{\boldsymbol{\Gamma}}{x^2} + \frac{\lambda_s}{x^\beta}\right)w(x) = 0$$

(26)

Whose general solution is:

$$w_I(x) = x^{\frac{1-g}{2}}\left(C_1 J_p\left(\frac{2\sqrt{\lambda_s}}{2-\beta}x^{1-\beta}/_2\right) + C_2 Y_p\left(\frac{2\sqrt{\lambda_s}}{2-\beta}x^{1-\beta}/_2\right)\right)$$

(27)

Where : $p = \frac{\sqrt{(1-g)^2-4\boldsymbol{\Gamma}}}{2-\beta}$and $J_p(x)$ and $Y_p(x)$are the well known Bessel's functions. Then the solution of the surface density in the region($\mathrm{x} < 1$)is $u(x) = \frac{w(x)}{q(x)}$ or

$$u_I(x) = x^{\frac{1-g}{2}}exp\left(-\frac{1}{2}\int^x f_1(z)dz + \frac{g}{2}\ln(x)\right)\left(C_1 J_p\left(\frac{2\sqrt{\lambda_s}}{2-\beta}x^{1-\beta}/_2\right) + C_2 Y_p\left(\frac{2\sqrt{\lambda_s}}{2-\beta}x^{1-\beta}/_2\right)\right)$$

(28)

Or after replacing $f_1(z)$ by its expression (4) and the torque by its expression (21), we find

$$u_I(x) = C_1 \frac{x^{-\frac{3}{4}-\beta}}{3CS^2}\left(\frac{2}{3}x + CS\sqrt{x}\right)\left(J_p\left(\frac{2\sqrt{\lambda_s}}{2-\beta}x^{1-\beta}/_2\right) + \frac{C_2}{C_1}Y_p\left(\frac{2\sqrt{\lambda_s}}{2-\beta}x^{1-\beta}/_2\right)\right)$$

(29)

By making the following considerations:

$$p = 1;\ \sqrt{(1-g)^2 - 4\boldsymbol{\Gamma}} = \frac{4}{3};\ \frac{C_2}{C_1} = -\frac{1}{10};\frac{C_1}{S} = 300; S = \frac{4}{15}; CS = -\frac{2}{3}, \beta = \frac{2}{3};\ \frac{2\sqrt{\lambda_s}}{2-\beta} = 9;\ \sqrt{\lambda_s} = 6 \quad (30)$$

Then the reduced $l(x)$ becomes: $\mathrm{l(x)} = \frac{3\mathrm{S}x^{\beta-1}}{3\mathrm{CS}+2\sqrt{x}} = \frac{3}{2}\frac{\mathrm{S}x^{-1/3}}{\sqrt{x}-1}$

$$l(x) = \frac{2}{5}\frac{x^{-1/3}}{\sqrt{x}-1}$$

(31)

Which is the torque that we have obtained, leading to the solution:

$$u_I(x) = -100.\,x^{-\frac{17}{12}}(x-\sqrt{x})\left(J\left(1,\ 9x^{\frac{2}{3}}\right) - \frac{Y(1{,}9x^{\frac{2}{3}})}{10}\right)$$

(32)

which is the surface density in our model that we have obtained and presented it in (Fig 1).

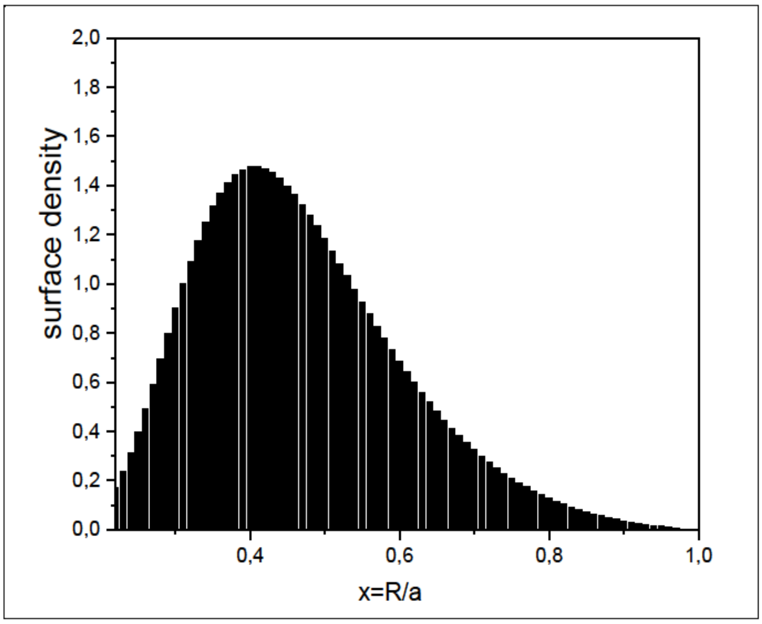

Figure 1: The surface density in region I.

3.2 IN THE REGIONI **R > a,x > 1**

Rewrite the general equation (4) as

$$f_2 u''(x)+f_1(x)u'(x)+f_0(x)u(x)=0 \quad (33)$$

Where:

$$f_0=\frac{3}{2x}\frac{v'(x)}{v(x)}+\frac{v''(x)}{v(x)}-\frac{l(x)}{\sqrt{x}v(x)}-\frac{2\sqrt{x}}{3v(x)}l'(x)+\frac{\lambda}{3v(x)} \quad (34)$$

$$f_1=\frac{3}{2x}+2\frac{v'(x)}{v(x)}-\frac{2\sqrt{x}}{3v(x)}l(x) \quad (35)$$

$$f_2=1 \quad (36)$$

And $v(x)=Sx^{\beta}$. If we put:

$$u(x)=w(x)\exp\left(\frac{1}{2}\int^{x}\frac{f_1(x')}{f_2(x')}dx'\right) \quad (37)$$

$$w''(x)+F(x)w(x)=0 \quad (38)$$

Such as

$$F(x)=\frac{f_0(x)}{f_2(x)}-\frac{1}{4}\left(\frac{f_1(x)}{f_2(x)}\right)^2-\frac{1}{2}\left(\frac{f_1(x)}{f_2(x)}\right)' \quad (39)$$

That is to say

$$F(x)=\frac{3}{16x^2}-\frac{l(x)}{3\sqrt{x}v(x)}-\frac{\sqrt{x}}{3v(x)}l'(x)+\frac{\lambda}{3v(x)}-\frac{x}{9v^2(x)}l^2(x)+\frac{v'(x)\sqrt{x}}{3v^2(x)}l(x) \quad (40)$$

By replacing the viscosity $v(x)=Sx^{\beta}$in the last expression of $F(x)$, we find

$$F(x)=\frac{3}{16x^2}-\frac{1}{3Sx^{\beta-\frac{1}{2}}}l'(x)+\frac{\lambda}{3Sx^{\beta}}-\frac{1}{9S^2x^{2\beta-1}}l^2(x)+\frac{(\beta-1)}{3S.x^{\beta+\frac{1}{2}}}l(x) \quad (41)$$

By using the same constraint on $l(x)$ as in the regionI

$$\frac{1}{x^{\beta-\frac{1}{2}}}l'(x)+\frac{1}{3Sx^{2\beta-1}}l^2(x)-\frac{(\beta-1)}{Sx^{\beta+\frac{1}{2}}}l(x)=0$$

(42)

$$l'(x)+\frac{1}{3Sx^{\beta-1/2}}l^2(x)-\frac{(\beta-1)}{x}l(x)=0$$

(43)

We find that

$$l(x)=\frac{3Sx^{\beta}}{2x^{3/2}+3CSx}$$

(44)

In this regionII, if we take $\beta=5/4$, $S=2/3$and $C=-0.97$, the corresponding $l(x)$ expression is

$$\mathrm{l(x)}=\frac{2\mathrm{x}^{5/4}}{2\mathrm{x}^{3/2}-1.94\mathrm{x}}$$

(45)

The solution in regionII is then given by

$$w_{II}=\sqrt{\frac{R}{a}}\left(K_1J_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}(x)^{3/8}\right)+K_1Y_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}(x)^{3/8}\right)\right)$$

(46)

$$u_{II}(x)=w_{II}(x)exp\left(-\frac{1}{2}\int^{x}\left(\frac{3}{2t}+2\frac{v\prime(t)}{v(t)}-\frac{2\sqrt{t}}{3v(t)}l(t)\right)dt\right)$$

(47)

Or after replacing w_{II} by (46) and the l(x) (45), we find the solution in this region II as

$$u_{II}(x)=\frac{3}{2}\left(\frac{1-1.0309\sqrt{x}}{x^{3/2}}\right)\left(K_1J_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}(x)^{3/8}\right)+K_2Y_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}(x)^{3/8}\right)\right)$$

(48)

The constants K_1 and K_2 must satisfy the boundary conditions:

$$u_{II}(1)=0$$

$$K_1J_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}\right)=-K_2Y_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}\right)$$

$$u_{II}(x)=\frac{3}{2}K_1\left(\frac{1-1.0309\sqrt{x}}{x^{3/2}}\right)J_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}\right)\left(\frac{J_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}(x)^{3/8}\right)}{J_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}\right)}-\frac{Y_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}(x)^{3/8}\right)}{Y_{\frac{2}{3}}\left(\frac{4\lambda\sqrt{2}}{3}\right)}\right)$$

(49)

If we take $\lambda = \frac{5}{6}K_2$ we obtain

$$u_{II}(x) = \frac{3}{2}K_1 J_{\frac{2}{3}}\left(\frac{10\sqrt{2}}{9}K^2\right)\left(\frac{1-1.0309\sqrt{x}}{x^{3/2}}\right)\left(\frac{J_{\frac{2}{3}}\left(\frac{10\sqrt{2}}{9}K^2 x^{3/8}\right)}{J_{\frac{2}{3}}\left(\frac{10\sqrt{2}}{9}K^2\right)} - \frac{Y_{\frac{2}{3}}\left(\frac{10\sqrt{2}}{9}K^2 x^{3/8}\right)}{Y_{\frac{2}{3}}\left(\frac{10\sqrt{2}}{9}K^2\right)}\right) \qquad (50)$$

Such that $J_{\frac{2}{3}}\left(\frac{10\sqrt{2}}{9}K^2\right) = 0.63209.$

For The mode $K = 1$ and for $K_1 < 0$, the surface density from our model, which correspond to regionII is presented in the fig2.

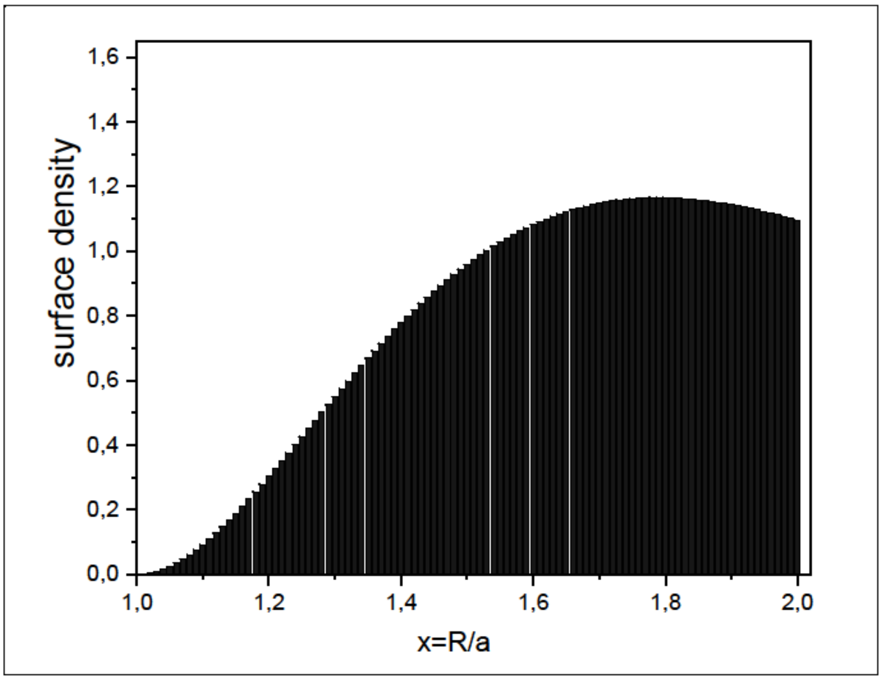

Figure 2 : The surface density in region II.

4. Results and Conclusion

In this paper, we describe the dynamics of a planet's gravitational interaction with a protoplanetary disk [1]. Figs. 1 and 2 show the behavior of the surface density , this expected behavior followed by the formation of a planet at (R = a) inside the disk. Through these curves, the density is very low near the planet (R = a), and it has a significant value away from the planet. The gap is dened as the region in which the surface density is less than of what it would be if there were no planet in the disk. This figure from our results correspond the approximation of [1]. We presented an overview of a giant planet migration in evolving protoplanetary disks.

The disks evolve as a result of the viscous transport of angular momentum, whereas planets migrate as a result of type II migration. We performed the calculations using an torque caused by the presence of a planet. We discovered solutions for surface density within a protoplanetary disk. This solution is depicted in (Figure 1 and Figure 2), the surface density is very low near the planet and has a significant value away from the planet.

Finally, we used the results of our solution of the evolution equation to contribute to the study of planetary migration. In this paper, we have obtained an overview of gaseous proto-planetary disk and embedded planet interactions. We found similar solutions in previous studies. What distinguishes our work is the solution of the surface density equation for the protoplanetary disk without using an angular momentum approximation. What the figures show in both regions demonstrates the validity of our work.

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