Coefficients Calculation of Laurent Series in the Neighborhood of Complex Variable Function Poles

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Abstract: The theory of complex function is a key part of mathematics, which can solve the complex problems in production and life. It is of great significance to extend the research field of complex function theory. In this paper, taking a complex variable function as the research object, a calculation method of Laurent series coefficient of complex function pole neighborhood expansion was proposed to determine the complex variable function pole, determine the order of complex variable function pole, calculate the residue of high-order pole in complex variable function, thus judging the attribute of complex variable function. In this regard, the coefficient formula was used to calculate the coefficients of Laurent series in the neighborhood of the complex variable function poles.
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1 Introduction

The theory of complex function is a branch of mathematics, which originated in the 18th century. As the pioneer of the theory of complex variable function, eulada Lambert and Laplace created the theory of complex variable function and studied the integration of complex variable function (he 2018). D'Alembert first proposed the theory of complex variable function in his paper on Fluid Mechanics (Wang et al. 2017). Then, Euler considers two equations derived from the integral of the function of complex variables. Therefore, these two equations are called " D'Alembert-Euler equation”. In the 19th century, Cauchy and Riemann studied these two equations in detail, so they are also called "Cauchy-Riemann conditions". After that, the research object of complex function is based on a class of analytic functions satisfying this condition in complex field [1].

The theory of function of complex variable has become an indispensable part of mathematics. It promotes the development of differential equation, integral equation, probability theory and number theory. Most of the modern mathematical theory comes from the mathematical research at that time. Complex function theory can solve many practical problems in many fields, such as physics and aeronautics. Nowadays, the basic content of complex function has become a compulsory course for science and engineering majors[2]. Mita-lefler, Poincare, Adama etc. have opened up the research field of complex function theory. Therefore, it is not only of theoretical value but also of practical significance to study the evolution of complex function theory [3].

Compared with foreign research status, the research on complex function in China is still in its infancy. Although there are a lot of discussions on the theory of complex functions, most of them are mainly reviews, and little research has been done on its evolution process. There is no monograph on the idea of Weierstras complex variable function in China [4]. The reason for this phenomenon is that the original materials are only kept in the archives of the German Academy of Sciences and the German University Library [5]. In addition, Weierstras manuscripts and publications are in German. The difficulty of literature acquisition and language barrier lead to the stagnation of research. Li et al. made use of literature review and language convenience to study the idea of complex variable function of Weierstras to make up for the lack of domestic research [6].

The theory of complex variable function has become one of the most favorable tools to solve practical problems, and it has practical effect [7]. For example, stable field calculation [8]; extensive use of fluid mechanics and aeronautical Mechanics [9]. Moreover, the expansion of the research field of complex function theory has made greater contributions to the development of this discipline in more fields [10]. In this work, the complex variable function is taken as the research object, which has a certain theoretical value for the history of discipline and practical value for practical problems.

2 Calculation for Coefficients of Laurent Series in the neighborhood of Poles of Complex Function

2.1 Judgment for Attribute of Function of Complex Variable

2.1.1 Determination for Poles of Complex Function

In this paper, a simple method was given to determine the poles of complex function such as $F(z)=\frac{f(z)}{g(z)}$. However, it was unnecessary to discuss the case of infinite points .

In general, the following facts have been clarified in complex functions,:

For the function $F(z)=\frac{f(z)}{g(z)}$:

(1)If $g(z)\neq 0$, $F(z)$ has nosingular point;

(2)If $g(a)=0$, $f(a)\neq 0$, when $z=a$ is the m-order zero point of $g(z)$, $z=a$ is the m-order zero point and n-order zero point of $F(z)$;

(3)If $g(a)=0$, and $f(a)=0$, $z=a$ is the common zero point of $f(a)$ and $g(z)$, which is not lower than the first order. When $z=a$, $z=a$ is also:

$$F(z)=\begin{cases} \text{Zeros of order } (m-n), & m>n \\ \text{Singularity can be removed}, & m=n \\ (n-m) \text{ order pole}, & m<n \end{cases} \quad (1)$$

On this basis, it is not difficult to prove the following conclusion.

Theorem 1: let $f(a)$ and $g(z)$ be resolved, and $z=a$ is the common zero point of $f(a)$ and $g(z)$, which is not lower than the first order. Then, when $z=a$ are poles (zero points) of $\frac{f(z)}{g(z)}$ and $\frac{f'(z)}{g'(z)}$, the orders are the same [11,12].

Expound and prove: $F(z)$ and $g(z)$ are resolved, and $f(a)=g(a)=0$. Let $f(z)=(z-a)^{a}\varphi(z)$, $(a\geq 1)$, $g(z)=(z-a)^{\beta}\phi(z)$, $(\beta\geq 1)$. Where, $\varphi(z)$ and $\phi(z)$ are resolved, and $\varphi(a)\neq 0$, $\varphi(a)\neq 0$, and then:

$$\frac{f(z)}{g(z)}=(z-a)^{a-\beta}\frac{\varphi(z)}{\phi(z)},\frac{f'(z)}{g'(z)}=(z-a)^{a-\beta}\frac{a\varphi(z)+(z-a)\varphi'(z)}{\beta\varphi(z)+(z-a)\phi'(z)} \quad (2)$$

$\phi(a)\neq 0$, so $\left[\beta\phi(z)+(z-a)\phi'(z)\right]\neq 0$.

When $\beta-a=m\geq 1$, $(a-\beta=m\geq 1)$, $z=a$ is also the m-order pole (zero point) of $\frac{f(z)}{g(z)}$ and $\frac{f'(z)}{g'(z)}$.

If we have noticed $a=\beta$ in the above proof, we can obtain:

Theorem 2: let $f(z)$ and $g(z)$ be resolved, and $f(a)=g(a)=0$, $g'(a)\neq 0$, then $z=a$ is the movable singularity of $\frac{f(z)}{g(z)}$.

Theorem 3:let $f(z)$ and $g(z)$ be resolved, and $f(a)=g(a)=0$, $f'(a)\neq 0$, then $z=a$ is the m-order pole of $\frac{f'(z)}{g'(z)}$. $z=a$ is the m-order pole of $\frac{f(z)}{g(z)}$.

Expound and prove: suppose that $z=a$ is the first-order zero point of $f(z)$, it is also m+1-order zero point of $g(z)$.

$\therefore$ is the m-order pole of $\frac{f(z)}{g(z)}$.

Inference: let $f(z)$ and $g(z)$ be resolved, and $f(a)=g(a)=0$. If $f^{(k)}(a)=0$, $f^{(k)}(a)\neq 0$, and $z=a$ is the m-order pole of $\frac{f^{(a)}(z)}{g^{(n)}(z)}$, then $z=a$ is the m-order pole of $\frac{f(z)}{g(z)}$.

Expound and prove: suppose that $z=a$ is the n-order zero point of $f(z)$, and it is also the m+n order zero point of $g(z)$. ∴ is the m-order zero point of $\frac{f(z)}{g(z)}$.

2.1.2 Method to Determine the Number of Orders of Polar Point of Complex Function

The function is simply expressed in three forms: quotient, sum and product: $f(z)=P(z)/Q(z)$, $f(z)=P(z)+Q(z)$ and $f(z)=P(z)Q(z)$. Compared with the form of expressing a function as $f(z)=1/g(z)$, this method is easier [13]. Moreover, the property of zero point and pole of complex function and the relationship between the zero point and pole are used to determine the number of orders of poles of the three functions .

(1) First case

$z=z_0$ is the m-order zero point of function $P(z)$, and it is also the n-order zero point of function $Q(z)$. Function $f_1(z)=P(z)/Q(z)$, $f_2(z)=P(z)+Q(z)$, $f_3(z)=P(z)Q(z)$. What is the relationship between z_0 and these functions?

z_0 is the m-order zero point of function $P(z)$, and it is also the n-order zero point of $Q(z)$. According to the definition of zero point, $P(z)$ and $Q(z)$ can be expressed as: $P(z)=(z-z_0)^m\varphi(z)$ and $Q(z)=(z-z_0)^n\phi(z)$. In addition, $\phi(z)$ and $\phi(z)$ are resolved in z_0. Meanwhile, $\phi(z_0)\neq 0$ and $\varphi(z_0)\neq 0$. m and N are integers.

$$f_1(z)=\frac{P(z)}{Q(z)}=\frac{(z-z_0)^m\varphi(z)}{(z-z_0)^n\phi(z)}=\frac{(z-z_0)^m}{(z-z_0)^n}\cdot\frac{\varphi(z)}{\phi(z)} \quad (3)$$

$\varphi(z_0)$ and $\phi(z_0)$ are resolved. According to the properties of analytic functions, when the denominator is not zero, the quotient of the two analytic functions is still an analytic function. $\varphi(z)\neq 0$, so $\frac{\varphi(z)}{\phi(z)}$ is resolved in the neighborhood of z_0.In the neighborhood of z_0, $\frac{\varphi(z)}{\phi(z)}$ is expanded into Taylor series.

$$\frac{\varphi(z)}{\phi(z)}=\sum_{n=0}^{\infty}c_n(z-z_0)^n=c_0+c_1(z-z_0)+c_2(z-z_0)^2+\dots \quad (4)$$

When $z = z_0$, $\frac{\varphi(z)}{\phi(z)} = c_0 \neq 0$.

When $m < n$,

$f_l(z) = \frac{1}{(z-z_0)^{n-m}} \cdot \frac{\varphi(z)}{\phi(z)} = \frac{c_0}{(z-z_0)^{n-m}} + \frac{c_1}{(z-z_0)^{n-m+1}} + \ldots + \frac{c_{n-m-1}}{z-z_0} + c_{n-m} + \ldots$

After $f_l(z)$ is expanded into Laurent series in the deleted neighborhood of z_0, the negative power terms of $(z-z_0)$ are finite. The highest power of term $(z-z_0)^{-1}$ is $(z-z_0)^{-(n-m)}$, and the coefficient of this term is c_0. $c_0 \neq 0$, and $m-n<0$ so z_0 is the pole point of $n-m$ order of $f_l(z)$.

When $m > n$, $f_l(z) = (z-z_0)^{m-n} \varphi(z)/\phi(z)$, and $\frac{\varphi(z)}{\phi(z)}$ is resolved within z_0. Meanwhile, $\frac{\varphi(z)}{\phi(z)} = c_0 \neq 0$ and $m-n>0$, so Laurent expansion on the deleted neighborhood of z_0 shown is as follows:

$$f_l(z) = \frac{1}{(z-z_0)^{m-n}} \cdot \varphi(z)/\phi(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots \quad (5)$$

In the formula, there is no negative power term of $(z-z_0)$, so z_0 is a movable singularity of $f_l(z)$.

$$f_2(z) = P(z) + Q(z) = (z-z_0)^m \varphi(z) + (z-z_0)^n \phi(z) \quad (6)$$

z_0 is the zero point of $f_2(z)$. When $\varphi(z)+\phi(z) \neq 0$, the number of orders of zero point is the smaller value in m and n. After the series expansion on the right of formula, the minimum power of the non-zero coefficient $(z-z_0)$ is the smaller value in m and n.

$$f_3(z) = P(z) \cdot Q(z) = (z-z_0)^{m+n} \varphi(z)\phi(z) \quad (7)$$

z_0 is the m+n order zero point of $f_3(z)$.

(2) Second situation

$z = z_0$ is the m-order pole of function $P(z)$, and it is also the n-order pole of function $Q(z)$. Function $f_1(z) = P(z)/Q(z)$, $f_2(z) = P(z)+Q(z)$, $f_3(z) = P(z) \cdot Q(z)$. z_0 is the m-order pole of function $P(z)$, and it is also the n-order pole of function $Q(z)$. According to the definition of poles, the following conclusions can be drawn.

$$P(z) = \frac{1}{(z-z_0)^m} \varphi(z), Q(z) = \frac{1}{(z-z_0)^n} \varphi(z) \quad (8)$$

Where, $\varphi(z)$ and $\phi(z)$ are analytic functions in z_0 -neighborhood, and $\varphi(z_0)\neq 0$, $\phi(z_0)\neq 0$.

$$f_1(z)=P(z)/Q(z)=\frac{(z-z_0)^n}{(z-z_0)^m}\cdot\frac{\varphi(z)}{\phi(z)} \tag{9}$$

When $m<n$:

$$f_1(z)=P(z)/Q(z)=\frac{(z-z_0)^n}{(z-z_0)^m}\cdot\frac{\varphi(z)}{\phi(z)}=(z-z_0)^{n-m}\cdot\frac{\varphi(z)}{\phi(z)}\cdot\frac{\varphi(z)}{\phi(z)} \tag{10}$$

It is resolved in the z_0 -neighborhood, and $\frac{\varphi(z)}{\phi(z)}\neq 0$, $n-m>0$. z_0 is the n-m order zero point of $f_1(z)$.

When $m>n$:

$$f_1(z)=P(z)/Q(z)=\frac{(z-z_0)^n}{(z-z_0)^m}\cdot\frac{\varphi(z)}{\phi(z)}\cdot\frac{1}{(z-z_0)^{n-m}}\cdot\frac{\varphi(z)}{\phi(z)} \tag{11}$$

$\frac{\varphi(z)}{\phi(z)}$ is resolved within z_0 -neighborhood, and $\frac{\varphi(z)}{\phi(z)}\neq 0$, $m-n>0$. z_0 is the n-m order pole of $f_1(z)$.

When $m=n$, z_0 is a movable singularity of $f_1(z)$.

$$f_2(z)=P(z)+Q(z) \tag{12}$$

When $m\neq n$, let $N=\max(m,n)$. z_0 is the n-order pole of $f_2(z)$. After Laurent expansion is applied to $P(z)$ and $Q(z)$, we add them up. The highest power of $(z-z_0)^{-1}$ can take the larger term in m and n.

When $m=n$ and $\varphi(z_0)+\phi(z_0)\neq 0$:

$$f_3(z)=P(z)+Q(z)=\frac{\varphi(z)+\phi(z)}{(z-z_0)^M},(N=m=n) \tag{13}$$

Therefore, z_0 is the m+n order pole point of $f_3(z)$.

In conclusions, the function with complex form can be expressed in simple form of function such as quotient, sum and product, and then the number of orders can be determined by the direct relationship between zero point and pole point of complex function .

2.1.3 Calculation of High-Order Pole Residues in Complex Function

The key of complex function is residue theorem, which is the combination of complex function series and integral. Therefore, there is an inseparable relationship between complex variable function and residue theorem .

According to the residue theorem, if function $f(z)$ has finite isolated singularity z_k in region $\bar{D}$, the analytic formula:

$$\int_L f(z)\,dz = 2\pi i \sum_k \operatorname{Res} f(b_k) \tag{14}$$

In the formula, the coefficient of Laurent series of $f(z)$ is represented by $\operatorname{Res} f(b_k)$ in the range of centerless neighborhood of b_k. The neighborhood range is $0<|z-b_k|<R$. $\operatorname{Res} f(b_k)$ is called the residue of $f(z)$ in $z=b_k$.

(1) If the complex variable function calculates the analytic function in the integral loop L, there is no singular point within L, and the residue of complex variable function is zero . According to the residue theorem in Formula (14), it can be described as follows:

$$\int_L f(z)\,dz = 0 \tag{15}$$

The above description is Cauchy theorem of the integration of complex function. The integral of analytic functions along the loop is zero in the range of single-pass region.

(2) If a complex function has a first-order pole in the integral loop L, the integral $\int_L \frac{f(z)}{1-a}dz$ is investigated. Where, a denotes the point in integral loop L. $z=a$ is called the first-order pole of complex function. According to the residue theorem in Formula (14) and the formula of calculating first-order pole residue, we can get:

$$\int_L \frac{f(z)}{1-a}dz = 2\pi i \operatorname{Res} f(a) = 2\pi i \lim_{z\to a}\left[(z-a)\frac{f(z)}{z-a}\right] = 2\pi i f \tag{16}$$

It can be simplified as follows:

$$f(a) = \frac{1}{2\pi i}\int_L \frac{f(z)}{z-a}dz \tag{17}$$

The above formula describes the high-order derivative formula of integral of complex variable function.

Based on the above formula, we can see that Cauchy theorem is the residue theorem of analytic function in the integral range ; Cauchy formula is the residue theorem that complex function has first-order poles within integral range. The high-order derivative formula is the residue theorem that the complex function has n+1 poles within integral range.

For the calculation of residues at the poles of complex functions, we can use the above theory to give the following lemma, and improve the calculation method.

Lemma 1: a is the m-level pole of $f(z)$, and the residue of $f(z)$ at point a is described as follows:

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right] \tag{18}$$

Lemma 2: a is the first-order point of $f(z)=\frac{\varphi(z)}{\phi(z)}$. $\varphi(z)$ and $\phi(z)$ are solved under the condition of a -order point, $\varphi(a)\neq 0$, and $\phi(a)\neq 0$, then $\operatorname*{Res}_{z=a} f(z) = \frac{\varphi(a)}{\phi(a)}$.

Lemma 3: suppose that $f(z)$ has a certain number of isolated singular points (including infinite points) in the extended z-plane, $a_1, a_2, ..., a_n$. ∞ shows that the sum of residues of $f(z)$ at each point is 0, then:

$$\operatorname*{Res}_{z=a} f(z) + \sum_{k=1}^{n} \operatorname{Res} f(z) = 0 \quad (19)$$

Through practical application, the above concepts and theorems can be used to effectively calculate the high-order pole residues in complex variable functions [14].

2.2 Coefficient Formula of Laurent Series of Complex Function in the Neighborhood of Pole Point

Theorem: if z_0 is the m-order pole of $f(z)$, in the centreless neighborhood $0 < |z - z_0| < \delta$ of z_0:

$$f(z) = \sum_{n=-m}^{+\infty} C_n (z - z_0)^n \quad (20)$$

In the formula, $C_n = \frac{1}{(m+n)!} \lim_{z \to z_U} \frac{d^{m+n}}{dz^{m+n}} \left[(z - z_0)^m f(z) \right]$.

Expound and prove: because $f(z) = C_{-m}(z-z_0)^{-m} + ... + C_{-1}(z-z_0)^{-1} + C_0 + C_1(z-z_0) + ... + C_n(z-z_0)^n + C_{n-1}(z-z_0)^{n+1}$, after multiplying the two ends of the formula, we can get:

$$(z-z_0)^m f(z) = C_{-m} + ... + C_{-1}(z-z_0)^{m-1} + C_0(z-z_0)^m + ... + C_0(z-z_0)^{m+n} + C_{n+1}(z-z_0)^{m-n} + ... \quad (21)$$

$m+n$ orderderivatives are calculated, and then we can see that $\frac{d^{m+n}}{dz^{m+n}} \left[(z - z_0)^m f(z) \right] = (m+n)! C_n + \}$ includes term of positive power of $z - z_0$.

When $z \to z_0$, We calculate the limit values at both ends. The limit value on the right is $(m+n)! C_n$. And then,

$$C_n = \frac{1}{(m+n)!} \lim_{z \to z_0} \frac{d^{m+n}}{dz^{m+n}} \left[(z - z_0)^m f(z) \right] \quad (22)$$

[QED]

If $n = -1$ in Formula (21), $\operatorname{Res}\left[f(z), z_0 \right] \lim_{z \to z_0} \frac{d^{m+n}}{dz^{m+n}} \left[(z - z_0)^m f(z) \right]$. This formula is the residue formula at the m-order pole which exists in general complex variable function.

In Formula (21), if $n = -m$, $C_{-m} = \lim_{z \to z_0} (z - z_0)^m f(z)$. With the increase of n, the number of taking the derivative is also increased. In this way, the Laurent coefficients can be obtained successively, and then Laurent expansion can be obtained[15]. In this paper, direct expansion method and Formula (21) are used to solve the Laurent expansion of function.

Example 1: expand $\frac{1}{\sin z}$ into Laurent series within $0 < |z < \pi$.

Solution: $\frac{1}{\sin z}$ is solved in $0 < |z < \pi$, it can be expanded into Laurent series in this deleted neighborhood [16].

$z=0$ is the first order pole of $\frac{1}{\sin z}$, so $m=1$.

According to Formula (21), we can see that

$$C_{-1}=\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}z\cdot\frac{1}{\sin z}=1$$,

$$C_0=\lim_{z\to z_0}\frac{d}{dz}\left[\left((z-z_0)f(z)\right)\right]$$
$$=\lim_{z\to z_0}\frac{d}{dz}\left(\frac{z}{\sin z}\right)=\lim_{z\to z_0}\frac{\sin z-z\cos z}{\sin^2 z}\underline{\underline{\text{Law of lobida}}}\lim_{z\to z_0}\frac{\cos z-\cos z+z\sin z}{2\sin z\cdot\cos z}=0$$

,

$$C_1=\frac{1}{2!}\lim_{z\to z_0}\frac{d^2}{dz^2}\left[(z-z_0)f(z)\right]=\frac{1}{2!}\lim_{z\to z_0}\frac{d^2}{dz^2}\left(\frac{z}{\sin z}\right)$$
$$=\frac{1}{2!}\lim_{z\to z_0}\frac{z\sin^3 z-2(\sin z-z\cos z)\sin z\cos z}{\sin^2 z}$$
$$=\frac{1}{2}\left(1-\frac{2}{3}\right)=\frac{1}{6}$$

.

We can also get $C_2=0$, $C_3=\frac{7}{360}$, so $\frac{1}{\sin z}=\frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+\ldots(0<|z<\pi)$. If the indirect expansion method is adopted, then:

$$\frac{1}{\sin z}=\frac{1}{z-\frac{z^3}{3!}+\frac{z^5}{5!}}=\frac{1}{z\left(1-\frac{z^2}{6}+\frac{z^2}{6}-\ldots\right)}$$
$$=\frac{1}{z}\left(1+\frac{z^2}{6}+\frac{7z^4}{360}+\ldots\right)$$
$$=\frac{1}{z}+\frac{z}{6}\frac{7z^4}{360}+\ldots$$

(23)

For this problem, the direct method is more complicated. For those functions whose higher derivative is expressed by general expression, we can use Formula (21) to solve Laurent expansion. The advantage are obvious [17].

Example 2: in $0<|z<1$, we can expand $\frac{e^z}{z^2(z+1)}$ into Laurent series.

Firstly, the indirect method is used toexpand

Solution 1:

$$\frac{e^z}{z^2(z+1)}=\frac{1}{z^2}\frac{1+z+\frac{z^2}{2!}+\ldots}{1+z}$$
$$=\frac{1}{z^2}\left(1+\frac{z^2}{2}-\frac{z^3}{3}-\frac{9}{24}z^4+\ldots\right)$$
$$=\frac{1}{z^2}+\frac{1}{2}-\frac{1}{3}z+\frac{9}{24}z^2+\ldots$$

(24)

If Formula (21) is used to expand directly, we can find the general term of Laurent series.

Solution 2:

$z=0$ is a secondary pole, so $m=2$. According to Formula(21), $C_{-2}=\lim_{z\to 0}z^2f(z)=\lim_{z\to 0}\frac{e^z}{z+1}=1$. Thus:

$$C_n=\frac{1}{(n+2)!}\lim_{z\to 0}\frac{d^{n+2}}{dz^{n+2}}\left[z^2f(z)\right]$$
$$=\frac{1}{(n+2)!}\lim_{z\to 0}\frac{d^{n+2}}{dz^{n+2}}\left(\frac{e^z}{z+1}\right)$$
$$(n=-1,0,1,2,\ldots)$$

(25)

Therefore,

$$C_n=\frac{1}{(n+2)!}\left(\frac{e^z}{z+1}\right)^{|n+2|}\Big|_{z=0}$$
$$=\frac{1}{(n+2)!}\left[\sum_{k=0}^{n+2}C_{n-2}^{k}\left(e^z\right)^{(n+2-k)}\left(\frac{1}{z+1}\right)^{\{k\}}\right]\Bigg\}_{z=0}$$
$$=\frac{1}{(n+2)!}\left[\sum_{k=0}^{n+2}C_{n-2}^{k}e^z\frac{(-1)^k k!}{(z+1)^{k-1}}\right]\Bigg\}_{z-0}$$
$$=\frac{1}{(n+2)!}\left[\sum_{k=0}^{n+2}C_{n-2}^{k}(-1)^k k!\right]$$
$$=\sum_{k=11}^{n+2}\frac{(-1)^k}{(n+2-k)!}$$
$$=\frac{1}{(n+2)!}-\frac{1}{(n+1)!}+\frac{1}{n!}-\ldots$$
$$+\frac{(-1)^{n+1}}{1!}+(-1)^{n+2}$$

(26)

After introducing $n=-1$, $n=0$, $n=1$ and $n=2$ into the formula, we can get

$C_{-1}=\frac{1}{1!}-1=0$, $C_0=\frac{1}{2!}-\frac{1}{1!}+1=\frac{1}{2}$,

$C_1=\frac{1}{3!}-\frac{1}{2!}+\frac{1}{1!}-1=-\frac{1}{3}$,

$C_2=\frac{1}{4!}-\frac{1}{3!}+\frac{1}{2!}-\frac{1}{1!}+1=-\frac{9}{24}$, then

$$\frac{e^z}{z^2(z+1)}=\frac{1}{z^2}+\frac{1}{2}-\frac{1}{3}z+\frac{9}{24}+\ldots+\left[\frac{1}{(n+2)!}-\frac{1}{(n+1)!}+\ldots+\frac{(-1)^{n+1}}{1!}+(-1)^{n+2}\right].$$

It is important to mention that if this problem is expanded by indirect method or two seriesmultiplied by diagonal method, the general term of series can also be obtained [18]. Some calculation skills are very important. The solving process is shown as follows:

Solution 3:

$$\frac{e^z}{z^2(z+1)}=\frac{1}{z^2}\left[\sum_{k=0}^{\infty}(-1)^e z^e\right]\left(\sum_{k=1}^{\infty}\frac{z^k}{\text{k!}}\right)$$
$$\underline{\underline{\text{Let } k+l=m}}\frac{1}{z^2}\sum_{k=0}^{\infty}z^m\left[\sum_{k=1}^{m}\frac{(-1)^e}{\text{k!}}\right]$$
$$=\sum_{k=0}^{\infty}z^{m-2}\left[\sum_{k=0}^{\infty}z^m\frac{(-1)^{m-k}}{\text{k!}}\right]$$
$$\underline{\underline{令 m-2=n}}\sum_{n=-2}^{\infty}z^m\left[\sum_{k=1}^{n+2}\frac{(-1)^{e-k}}{\text{k!}}\right]$$

(27)

When the integral path C is an unclosed curve, the solution strategy is:

If the complex variable function $f(z)$ is not solved in region D, $\int_C f(z)dz$ is related to the integral path. According to the condition of complex integral, the parameter method is adopted . The specific steps are shown as follows:

(1) Write the parameter equation of integral path C

$$z=z(t)$$

(28)

(2) $z=z(t)$ is introduced $f(z)$ into and dz, we get the following results:

$$f(z)=f(z(t)),dz=z'(t)dt$$

(29)

(3) According to the second step, we can get $\int_C f(z)dz=\int_{\infty}^{\beta}f(z(t)z'(t))dt$. The integral of real parameter t on the right of

the formula is calculated.

The parameter method is also a method to solve the integral of complex function [19].

It is necessary to calculate the unit semicircle circumference of the left-half plane of $\int_C |z| dz, C$ with the origin as the center in the counter clockwise direction [20].

Solution: Integral path C is an unclosed curve. The complex variable function $f(z)=|z|$ is not analyzed in the complex plane, so the parameter method is adopted.

Let $z=e^{i\theta}$, $2\pi \le \theta \le \frac{3}{2}\pi$, and then $dz = ie^{i\theta}d\theta$, $f(z)=|z|=|e^{i\theta}|=1$. Thus:

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = F(z_2)-F(z_1) \tag{30}$$

This method is called the original function method [21].

When the integral path C is a closed curve, the solution strategy is:

Function $f(z)$ is resolved in region D, and C is included in D. According to Cauchy integral basic theorem, $\int_C f(z)dz = 0$.

Example 1: calculate the value of integral $\int_{|z|=S} (2z^2+e^3+\cos z)dz$.

Solution: $2z^2$, e^3 and $\cos z$ are solved on complex plane, their sum is analytic in the connected region including $|z|=5$. Therefore, $\int_{|z|=S} (2z^2+e^3+\cos z)dz = 0$.

Function $f(z)$ is solved on the region D + C, where C is the boundary of D, then:

$$\int_c f(z)dz = 0 \tag{31}$$

Example 2: calculating the value of integral $\int_{|z+1|=\frac{1}{4}} \frac{3z}{(z-2)(z-1)}dz$.

Solution: Region G: $|z+1| \le \frac{1}{4}$, and obvious C: $|z+1| = \frac{1}{4}$ is the boundary of G, and $f(z)$ is solved on G + C. According to the basic theorem of Cauchy integral, we can get:

$$\int_{|z+1|=\frac{1}{4}} \frac{3z}{(z-2)(z-1)}dz = 0 \tag{32}$$

The complex function $f(z)$ has finite singularities in D. The method of digging singularity is adopted [22]. Then:

$$\int_c f(z)dz = \sum_{i=1}^{n} \int_C f(z)dz \text{ conversion}$$
$$\begin{cases} \int_C f(z) = \iint_{C_i} f(z)dz = 2\pi i g(z_i) (\text{cauchy integral}) \\ \int_C f(z) = \iint_{C_i} f(z)dz = 2n! (\text{Higher order derivative}) \end{cases} \tag{33}$$

According to the compound closed-circuit theorem, the singularity digging method uses some non-overlapping closed curves to include the singular points in integral path C in a closed curve (these closed curves are included in C), and then the complex integral is calculated by Cauchy integral formula and higher-order derivative formula [23].

3 Conclusions

In the 19th century, the theory of complex function has been developed in an all-round way and dominated the field of mathematics in the 19th century. Cauchy, Weierstraz and Riemann devoted themselves to the basic research of function theory of complex variables, and gave the subject formal characteristics. At that time, mathematicians recognized that the theory of complex variable function was the most abundant branch of mathematics, enjoying the fun of studying the theory of complex variable function. Klein praised it as one of the most harmonious theories in abstract science. At present, there is little research on the calculation of coefficient of Laurent series in the neighborhood of the poles of complex variable function. In this paper, the determination method of its attribute and the method of solving coefficient are analyzed, which provides guidance for the study of complex function in the future.

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