

# C-2 Continuous Cubic Hermite Interpolation Splines with Second-Order Elliptic Variation

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**Abstract:** In order to solve the deficiency of Hermite interpolation spline with second-order elliptic variation in shape control and continuity, c-2 continuous cubic Hermite interpolation spline with second-order elliptic variation was designed. A set of cubic Hermite basis functions with two parameters was constructed. According to this set of basis functions, the three-order Hermite interpolation spline curves were defined in segments C~2, and the parameter selection scheme was discussed. The corresponding cubic Hermite interpolation spline function was studied, and the method to determine the residual term and the best interpolation function was given. The results of an example show that when the interpolation conditions remain unchanged, the cubic Hermite interpolation spline curves not only reach C~2 continuity, but also can use the parameters to control the shape of the curves locally or globally. By determining the best values of the parameters, the cubic Hermite interpolation spline function can get a better interpolation effect, and the smoothness of the interpolation spline curve is the best.

**Keywords:** Second order elliptic variation; C ~ 2 continuous; Cubic trigonometry; Interpolation spline; Hermite basis function

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## 1 Introduction

Computer aided geometric design (CAGD) is an emerging interdisciplinary involves mathematics and computer science, with the rapid development of science and

technology and the widespread use of computer, the emergence of the computer aided geometric design not only promote the development of the world industry, but also to the development of modern industry has a

very big influence. The term CAD was first coined by Barnhill and Riesenfeld at the university of Utah and later developed by Bezier and Coons in the 1960s. The main research object of computer-aided geometric design is the construction of curves and surfaces and their shape control [1]. The main purpose of its research is to solve the mathematical description of the shape of industrial products. In many industrial products, such as: aircraft, automobile and ship shape design, engineering device mold design, biomedical image processing, etc. need to use computer-aided geometric design.

The shape design of free curve and surface is the core problem in computer aided geometry design. It was proposed by Bezier and Coons et al in the 1960s and developed by scholars of various countries. In 1963, Ferguson of Boeing corporation first proposed the method of expressing curve and surface as vector function of parameters. Since then, the parameterized form of curve and surface has become the standard form of shape mathematical description. In 1964, Massachusetts institute of technology (MIT) professor Coons proposed a method to describe surface. In 1967, Coons further described this surface in detail, making it widely used in computer-aided geometric design. In 1971, Bezier, a French Renault company, proposed a new method to design curves by controlling polygons. By controlling vertices, the shape of curves can be changed. Bezier method is not only simple and practical, but also solves the shape control problem of curves and surfaces. In 1974,

Gordon of general electric laboratory and Riesenfeld of the university of Utah combined the control polygon method with the Hermite interpolation spline theory, and finally proposed the Hermite interpolation spline method. In 1975, in order to overcome the problem that the Hermite interpolation spline curve could not accurately represent the quadric curve surface, Versprille first mentioned the rational Hermite interpolation spline method in his doctoral thesis [2]. Then, Piegl, Tiller and Farin et al. proposed the non-uniform rational Hermite interpolation spline. Due to the limitations of NURBS method and the complexity of calculation, scholars at home and abroad began to look for other new methods to construct curves and surfaces, so the cubic Hermite interpolation spline attracted the attention of many scholars. As early as 1964, Schoenberg first proposed the cubic Hermite interpolation spline function. Then in 1967, Lyche, Winther and Schumaker established a recursive formula for the Hermite interpolation spline of arbitrary order. In recent years, many scholars at home and abroad have made researches on cubic Hermite interpolation spline function [3].

C-2 continuous cubic Hermite interpolation is one of the basic topics in computer-aided geometric design, and it is also a very active research field. The basic problem with the C-2 continuous cubic Hermite interpolation is to construct a curve (a surface) that strictly passes through a given set of type-value points (a set of three-dimensional data points generally measured from the surface of a product

model or an imitation). This kind of interpolation problem belongs to the category of Reverse Engineering and has been widely applied in automatic design, mechanical manufacturing, simulation and medicine.

C-2 continuous cubic Hermite interpolation spline is a very effective and widely used tool for curve and surface description in CAGD and CG [4]. But, usually the splines, such as engineering of the common standard cubic Hermite interpolation spline, determine the condition of interpolation, the shape of a spline curve is fully identified, changes in plants could also determine its approximation effect, have shape cannot be modified, can not accurately said some common defects such as engineering curve, the interpolation method called deterministic interpolation. How to flexibly control the shape of the spline curve under given interpolation conditions, so as to improve the approximation effect of the curve and make it more suitable for practical engineering, is an important research content in CAGD. In recent years, Hermite interpolation splines with rational form of parameters have aroused wide interest of many scholars. These spline curves not only have similar properties to the standard cubic Hermite interpolation spline, but also can control the shape of the interpolation spline by selecting appropriate shape adjustment parameters [5]. However, by analyzing the analytical expressions of these rational splines, we can see that their numerator and denominator are linear algebraic polynomials, so for some common quadratic curves in

engineering, we still can't get accurate expressions by using them. In order to achieve this goal, many scholars have studied the nonlinear spline curve to some extent. In various forms of generalization, splines containing trigonometric functions have attracted much attention because of their unique theoretical significance and practical application value. Many scholars have made many useful discussions on the spline curve based on trigonometric functions and obtained many useful results. Including Xie Jin etc, parameters associated with rational Hermite interpolation spline cannot accurately said curve the common defects in engineering, and puts forward a rational containing parameter three triangle Hermite interpolation spline, and the general rational spline, adjustment parameters can be used to control the shape of a spline curve, in addition, under certain conditions, the corresponding Ferguson spline curve can also said some of the common in the engineering curve accurately. However, since the expressions of these interpolation splines are all expressed in rational mathematical expressions, their expressions become very complex, leading to instability in the calculation, and the amount of calculation generated will increase greatly [6].

For this, this article adopts the method of constructing basis function, in standard cubic Hermite interpolation b-spline basis function, on the basis of trigonometric polynomial function is introduced, and the appropriate change, get a set of belt shape adjustable parameter of the three triangle Hermite

spline basis function, to generate a new type of interpolation spline curve, this paper called adjustable form three triangle Hermite interpolation spline curve [7]. The spline curve is similar to standard cubic Hermite interpolation spline in nature, in contrast, the trigonometric function is contained in the new expression of the spline interpolation polynomial, by selecting proper control points and shape adjustable parameter, the interpolation spline can accurate said straight line, arc, ellipse and free curve curve of common engineering [8]. In addition, because the cubic Hermite interpolation spline proposed in this paper avoids rational mathematical expression, its expression is relatively concise and requires relatively small amount of calculation, providing a new theoretical method for the design of interpolation curves and surfaces.

**2C-2 continuous cubic Hermite interpolation spline design for second-order elliptic variation**

**2.1 Establish C~2 continuous three times trigonometric Hermite basis function**

Definition 1: for  $0 \leq t \leq 1, \alpha, \beta \in R$ , the following four functions of the variable t:

$$\begin{cases} f_0(t) = a_0(t)\alpha + b_0(t) \\ f_1(t) = a_1(t)\alpha + b_1(t) \\ g_0(t) = c_0(t)\beta + d_0(t) \\ g_1(t) = c_1(t)\beta + d_1(t) \end{cases} \tag{1}$$

The above formula is called the c-2 trigonometric Hermite basis function of the second-order elliptic variation parameter, genus and genus. Among them,

$$\begin{cases} a_0(t) = t^2 - 10t^4 + 20t^5 - 15t^6 + 4t^7 \\ b_0(t) = 1 - 35t^4 + 84t^5 - 70t^6 + 20t^7 \\ a_1(t) = 5t^4 - 14t^5 + 13t^6 - 4t^7 \\ b_1(t) = 35t^4 - 84t^5 + 70t^6 - 20t^7 \\ c_0(t) = t^3 - 4t^4 + 6t^5 - 4t^6 + t^7 \\ d_0(t) = t - 20t^4 + 45t^5 - 36t^6 + 10t^7 \end{cases} \tag{2}$$

Theorem 1. C~2 continuous three times triangular Hermite basis function has the following properties:

Property 1: endpoints. The basis function satisfies at the endpoint:

$$\begin{pmatrix} f_0(0) & f_1(0) & g_0(0) & g_1(0) \\ f_0(1) & f_1(1) & g_0(1) & g_1(1) \\ f_0'(0) & f_1'(0) & g_0'(0) & g_1'(0) \\ f_0'(1) & f_1'(1) & g_0'(1) & g_1'(1) \\ f_0''(0) & f_1''(0) & g_0''(0) & g_1''(0) \\ f_0''(1) & f_1''(1) & g_0''(1) & g_1''(1) \\ f_0'''(0) & f_1'''(0) & g_0'''(0) & g_1'''(0) \\ f_0'''(1) & f_1'''(1) & g_0'''(1) & g_1'''(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\alpha & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \\ 0 & 0 & 2\alpha & 0 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix} \tag{3}$$

Property 2: quasi symmetry. The basis function satisfies:

$$\begin{cases} f_0(1-t) = f_1(t) \\ g_0(1-t) = -g_1(t) \end{cases} \tag{4}$$

Property 3: monotonicity. When the variable  $t \in [0,1]$  is fixed, the parameter  $f_i(t)$  ( $i=0,1$ ) increases monotonically;  $g_0(t)$  is about monotone increasing of parameter  $\beta$ ,  $g_1(t)$  is about monotone increasing of parameter  $\beta$ , and  $g_1(t)$  is about

monotone decreasing of parameter  $\beta$  [9].

It is proved that the endpoint property and quasi symmetry can be obtained by simple derivation. This paper only proves monotone.

When fixed variable  $t \in [0,1]$ , it can be obtained:

$$\frac{df_0}{d\alpha} = a_0(t) = t^2(1-t)^4(1+4t) \geq 0 \tag{5}$$

$$\frac{df_1}{d\alpha} = a_1(t) = t^4(1-t)^2(5-4t) \geq 0 \tag{6}$$

$$\frac{dg_0}{d\beta} = c_0(t) = t^3(1-t)^4 \geq 0 \tag{7}$$

$$\frac{dg_1}{d\beta} = c_1(t) = -t^4(1-t)^3 \geq 0 \tag{8}$$

Therefore,  $f_i(t)(i=0,1)$  is monotonically increasing with respect to  $\alpha$ ,  $g_0(t)$  is monotonically decreasing with respect to  $\beta$ , and  $g_1(t)$  is monotonically decreasing with respect to  $\beta$ .

## 2.2 Draw C~2 trigonometric Hermite parametric spline curves

### 2.2.1 Definition and properties of three consecutive triangular Hermite parametric spline curves for C~2

Based on the c-2 continuous three-times triangular Hermite basis function, the corresponding segmented c-2 continuous three-times triangular Hermite parameter spline curve can be defined.

Definition 2: given a list of type value points  $p_i$  and its tangent vector

$p'_i(i=0,1,\dots,n)$ , for  $0 \leq t \leq 1, \alpha, \beta \in R$ , the curve

$$r_i(t) = f_0(t)p_i + f_1(t)p_{i+1} + g_0(t)p'_i + g_1(t)p'_{i+1}$$

is called the cubic Hermite parameter spline curve with second-order ellipse becoming parameters  $\alpha$  and  $\beta$ .

Among them,  $f_j(t)$  and  $g_j(t) \quad j=0,1,\dots,n-1$  are three consecutive triangular Hermite basis functions of C~2 defined as 1.

Theorem 2: C~2 continuous cubic Hermite parametric spline curves have the following properties:

Property 1: interpolation. Hermite parameter spline curve was interpolated at the given value point  $p_i$  and its tangent vector  $p'_i(i=0,1,\dots,n)$ .

Property 2: continuity. The Hermite parametric spline curve is continuous from C to 2.

Property 3: symmetry. The Hermite parameter spline curve defined by type value point  $p_i$  and its tangent vector  $p'_i(i=0,1,\dots,n)$  is the same as the curve defined by type value point  $p_{n-i}$  and its tangent vector  $p'_i(i=0,1,\dots,n)$ , except that the parameterization direction is opposite [10].

Property 4: shape tunability.

When type value point  $p_i$  and tangent

vector  $p'_i(i=0,1,\dots,n)$  are fixed, the

shape of Hermite parameter spline curve can be adjusted as A whole by

the values of parameters  $\alpha$  and  $\beta$ .

Proof 1: by calculation:

$$\begin{cases} r_i(0) = p_i \\ r_i(1) = p_{i+1} \\ r'_i(0) = p'_i \\ r'_i(1) = p'_{i+1} \end{cases} \tag{9}$$

$$\begin{cases} r''_i(0) = 2\alpha p_i \\ r''_i(1) = 2\alpha p_{i+1} \\ r'''_i(0) = 6\beta p'_i \\ r'''_i(1) = 6\beta p'_{i+1} \end{cases} \tag{10}$$

Formula (9) shows that the three-order triangular Hermite parameter spline curves of C~2 are interpolated at the value point  $p_i$  and its tangent vector  $p'_i(i=0,1,\dots,n)$ .

Proof 2: when parameters  $\alpha$  and  $\beta$  are determined, it can be obtained from equations (9) and (10) that:

$$r_i^{(k)}(1) = r_{i+1}^{(k)}(0), k = 0, 1, 2, 3 \tag{11}$$

Formula (11) shows that the three-times continuous triangular Hermite parameter spline curve from C to 2 is continuous from C to 2.

Proof 3: from equations (3) and (4):

$$r_i(1-t) = f_1(t)p_i + f_0(t)p_{i+1} - g_1(t)p'_i - g_0(t)p'_{i+1} = R_i(t) \tag{12}$$

According to formula (2) and (12), it can be calculated as follows:

$$\begin{cases} R_i(0) = p_{i+1} \\ R_i(1) = p_i \\ R'_i(0) = p'_{i+1} \\ R'_i(1) = p'_i \\ R''_i(0) = 2\alpha p_{i+1} \\ R''_i(1) = 2\alpha p_i \\ R'''_i(0) = 6\beta p'_{i+1} \\ R'''_i(1) = 6\beta p'_i \end{cases} \tag{13}$$

Formula (13) shows that the C~2 continuous triangular Hermite parameter spline curve  $R_i(t)$  defined by type value point  $p_{n-i}$  and its tangent vector  $p'_{n-i}(i=0,1,\dots,n)$  is the same as the C~2 continuous triangular Hermite parameter spline curve  $r_i(t)$  defined by type value point  $p_i$  and its tangent vector  $p'_i(i=0,1,\dots,n)$ , except that the second order ellipse changes in the opposite parameterization direction.

Proof 4: according to equation (4), there are second-order elliptic variation parameters  $\alpha$  and  $\beta$ , so when the value point  $p_i$  and its tangent vector  $p'_i$  are fixed, the second-order elliptic variation parameter  $\alpha$  and  $\beta$  are taken with different values to

correspond to curves of different shapes, that is, the c-2 three-times continuous triangular Hermite parameter spline curve has shape adjustability [11].

Note 1. From theorem 2, it can be known that the c-2 continuous cubic Hermite parameter spline curve inherits the main characteristics of the classical cubic Hermite parameter spline curve, but it also has the following differences from the classical cubic Hermite parameter spline curve:

1. The classical cubic Hermite parameter spline curve only satisfies  $C^1$  continuity, while C~2 continuous cubic Hermite parameter spline curve can reach  $C^3$  continuity, so it is suitable for applications with high requirement for continuity.

2. When the type value points and tangent vector is fixed, the classic three triangle Hermite parameter spline curve shape will be fixed,  $C \sim 2$  continuous three trigonometric spline curve Hermite parameters could be changed by the second order elliptic shape parameters of  $\alpha$  and  $\beta$  for overall adjustment, so often in accord with the actual engineering needs to adjust the interpolation.

### 2.2.2 Second order elliptic variation parameter selection scheme

It can be known from the shape adjustability of the three-order triangular Hermite parameter spline curve from C to 2 that when the value point  $p_i$  and its tangent vector  $p'_i(i=0,1,\dots,n)$  remain unchanged, the interpolation curve can theoretically

have arbitrary shape by changing the values of parameters  $\alpha$  and  $\beta$  [12].

However, in many practical engineering problems, people often need to make the three-times continuous triangular Hermite parameter spline curve of C~2 meet the corresponding requirements according to a specific target, and then it is necessary to determine the values of parameters  $\alpha$  and  $\beta$  according to the given criteria. The following are three criteria for determining the values of parameters  $\alpha$  and  $\beta$ .

#### Rule 1. Shortest arc length

In general, the arc length of parametric curve  $r(t)(a \leq t \leq b)$  can be

expressed as  $L = \int_a^b |r'(t)| dt$ . Therefore,

after the stereotyped value point  $p_i$

and its tangent vector  $p'_i(i=0,1,\dots,n)$

were given, in order to make the triangular Hermite parameter

$r_i(t)(i=0,1,\dots,n-1)$  of C~2 have the

shortest arc length for three consecutive times, it was necessary to

determine the optimal values of

parameters  $\alpha$  and  $\beta$  to make the

arc length shortest, and then an

optimization model was established:

$$\begin{cases} \min L(\alpha, \beta) = \sum_{i=1}^{n-1} \int_0^1 |r'_i(t)| dt \\ s.t. \alpha, \beta \in R \end{cases} \tag{14}$$

For the convenience of calculation, equation (14) can be approximately equivalent to the following equation:

$$\begin{cases} \min L(\alpha, \beta) = \sum_{i=1}^{n-1} \int_0^1 (r'_i(t))^2 dt \\ \text{s.t. } \alpha, \beta \in R \end{cases} \quad (15)$$

According to equation (1), equation (4) can be rewritten as:

$$r_i(t) = L_i(t)\alpha + M_i(t)\beta + N_i(t) \quad (16)$$

Among them,

$$\begin{cases} L_i(t) = a_0(t)p_i + a_1(t)p_{i+1} \\ M_i(t) = c_0(t)p'_i + c_1(t)p'_{i+1} \\ N_i(t) = b_0(t)p_i + b_1(t)p_{i+1} + d_0(t)p'_i + d_1(t)p'_{i+1} \end{cases} \quad (17)$$

By substituting equation (17) into equation (16), the following formula can be obtained through calculation:

$$\begin{cases} \min L(\alpha, \beta) = a_1\alpha^2 + a_2\beta^2 + 2a_3\alpha\beta + 2a_4\alpha + 2a_5\beta + a_6 a_1 a_2 - a_3^2 = 0 \\ \text{s.t. } \alpha, \beta \in R \end{cases} \quad (18)$$

Where  $a_i (i=1,2,3,\dots,6)$  is a constant.

$$a_1 = \sum_{i=1}^{n-1} \int_0^1 (L'_i(t))^2 dt \quad (19)$$

$$a_2 = \sum_{i=1}^{n-1} \int_0^1 (M'_i(t))^2 dt \quad (20)$$

$$a_3 = \sum_{i=1}^{n-1} \int_0^1 (M'_i(t) \cdot L'_i(t)) dt \quad (21)$$

$$a_4 = \sum_{i=1}^{n-1} \int_0^1 (N'_i(t) \cdot L'_i(t)) dt \quad (22)$$

$$a_5 = \sum_{i=1}^{n-1} \int_0^1 (N'_i(t) \cdot M'_i(t)) dt \quad (23)$$

$$a_6 = \sum_{i=1}^{n-1} \int_0^1 (N'_i(t))^2 dt \quad (24)$$

For formula (18) to be true, there must be:

$$\begin{cases} \frac{\partial L(\alpha, \beta)}{\partial \alpha} = a_1\alpha + a_3\beta + a_4 = 0 \\ \frac{\partial L(\alpha, \beta)}{\partial \beta} = a_3\alpha + a_2\beta + a_5 = 0 \end{cases} \quad (25)$$

Then, the optimal value of second-order elliptic variation parameters can be obtained from equation (25) as follows:

$$\begin{cases} \alpha = \frac{a_3 a_5 - a_2 a_4}{a_1 a_2 - a_3^2} \\ \beta = \frac{a_3 a_4 - a_1 a_5}{a_1 a_2 - a_3^2} \end{cases} \quad (26)$$

Note 3. In practical application, if

solution to equation (25), and the value points of type can be adjusted appropriately to make  $a_1 a_2 - a_3^2 \neq 0$  true.

### Rule 2. Optimal smoothing

In general, the smoothness of second-order elliptic change parameter curve  $A r(t)(a \leq t \leq b)$  can be approximated by its energy value  $E = \int_a^b (r''(t))^2 dt$ , the smaller the energy value, the more smooth the curve [13]. Therefore, after the stereotyped value point  $p_i$  and its tangent vector

$p'_i (i=0,1,\dots,n)$  were given, the optimal

values of parameters  $\alpha$  and  $\beta$  needed to be determined to minimize their energy values in order to make the triangular Hermite parameter spline curve  $E r_i(t)(i=0,1,\dots,n-1)$  smoothed for three consecutive times from C to 2, and the optimization model was as follows:

$$\begin{cases} \min E(\alpha, \beta) = \sum_{i=1}^{n-1} \int_0^1 (r_i''(t))^2 dt \\ \text{s.t. } \alpha, \beta \in R \end{cases} \quad (27)$$

Substitute equation (16) into equation (27). After calculation, it can be obtained:

$$\begin{cases} \min L(\alpha, \beta) = b_1\alpha^2 + b_2\alpha^2 + 2b_3\alpha\beta + 2b_4\alpha + 2b_5\beta + b_6 \\ \text{s.t. } \alpha, \beta \in R \end{cases} \quad (28)$$

Where,  $b_i(i=1,2,3,\dots,6)$  is a constant.

$$b_1 = \sum_{i=1}^{n-1} \int_0^1 (L_i''(t))^2 dt \quad (29)$$

$$b_2 = \sum_{i=1}^{n-1} \int_0^1 (M_i''(t))^2 dt \quad (30)$$

$$b_3 = \sum_{i=1}^{n-1} \int_0^1 (M_i''(t) \cdot L_i''(t)) dt \quad (31)$$

$$b_4 = \sum_{i=1}^{n-1} \int_0^1 (N_i''(t) \cdot L_i''(t)) dt \quad (32)$$

$$b_5 = \sum_{i=1}^{n-1} \int_0^1 (N_i''(t) \cdot M_i''(t)) dt \quad (33)$$

$$b_6 = \sum_{i=1}^{n-1} \int_0^1 (N_i''(t))^2 dt \quad (34)$$

For equation (28) to be true, it must be:

$$\begin{cases} \frac{\partial E(\alpha, \beta)}{\partial \alpha} = b_1\alpha + b_3\beta + b_4 = 0 \\ \frac{\partial E(\alpha, \beta)}{\partial \beta} = b_3\alpha + b_2\beta + b_5 = 0 \end{cases} \quad (35)$$

When  $b_1b_2 - b_3^2 \neq 0$ , the optimal value of parameters can be obtained from equation (35):

$$\begin{cases} \alpha = \frac{b_3b_5 - b_2b_4}{b_1b_2 - b_3^2} \\ \beta = \frac{b_3b_4 - b_1b_5}{b_1b_2 - b_3^2} \end{cases} \quad (36)$$

Note 4. In practical application, if  $b_1b_2 - b_3^2 = 0$ , then there is no unique solution to equation (35), then the value points of type can be adjusted appropriately to make  $b_1b_2 - b_3^2 \neq 0$  true.

Rule 3. Give consideration to the shortest and smoothest arc length

When the stereotyped value point  $p_i$  and its tangent vector  $p_i'(i=0,1,\dots,n)$  are given, an optimization model can be established to make C~2 continuous three-order triangular Hermite interpolation curve  $r_i(t)(i=0,1,\dots,n-1)$  give consideration to both the shortest and smoothed arc length:

$$\begin{cases} \min F(\alpha, \beta) = L(\alpha, \beta) + E(\alpha, \beta) \\ \text{s.t. } \alpha, \beta \in R \end{cases} \quad (37)$$

To make equation (37) valid, then:

$$\begin{cases} \frac{\partial F(\alpha, \beta)}{\partial \alpha} = (a_1 + b_1)\alpha + (a_3 + b_3)\beta + (a_4 + b_4) = 0 \\ \frac{\partial F(\alpha, \beta)}{\partial \beta} = (a_3 + b_3)\alpha + (a_2 + b_2)\beta + (a_5 + b_5) = 0 \end{cases} \quad (38)$$

When

$(a_1 + b_1)(a_2 + b_2) - (a_3 + b_3)^2 \neq 0$  , the optimal value of parameters can be obtained from equation (38) :

$$\begin{cases} \alpha = \frac{(a_3 + b_3)(a_5 + b_5) - (a_2 + b_2)(a_4 + b_4)}{(a_1 + b_1)(a_2 + b_2) - (a_3 + b_3)^2} \\ \beta = \frac{(a_3 + b_3)(a_4 + b_4) - (a_1 + b_1)(a_5 + b_5)}{(a_1 + b_1)(a_2 + b_2) - (a_3 + b_3)^2} \end{cases} \quad (39)$$

In practical application, if  $(a_1 + b_1)(a_2 + b_2) - (a_3 + b_3)^2 = 0$ , then there is no unique solution to equation (39), then  $(a_1 + b_1)(a_2 + b_2) - (a_3 + b_3)^2 \neq 0$  can be established by appropriately adjusting value points.

### 3The example analysis

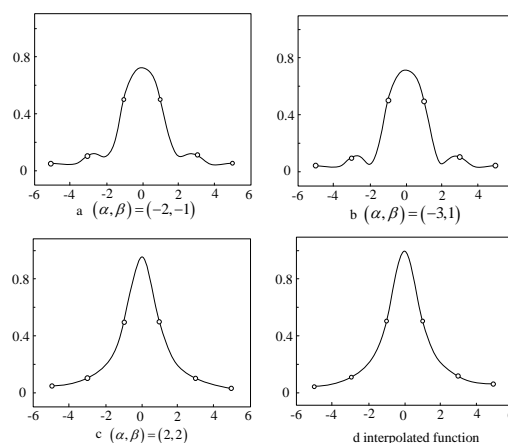
Example 1: suppose the interpolated function

$$y = f(x) = \frac{1}{1+x^2}, -5 \leq x \leq 5 \quad (40)$$

Take

$$x_i = -5 + 2i, y_i = f(x_i), m_i = f'(x_i) (i = 0, 1, \dots, n)$$

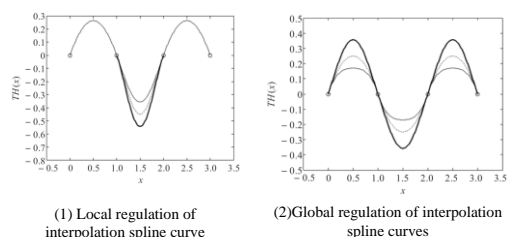
, and when parameters  $\alpha$  and  $\beta$  are taken with different values, the curves of c-2 continuous cubic Hermite interpolation function and interpolated function of different shapes are shown in figure 1.



**FIG. 1 C~2 continuous cubic Hermite interpolation function and interpolated function**

As can be seen from figure 1, for the interpolated function  $y = f(x) = \frac{1}{1+x^2}, -5 \leq x \leq 5$  , the interpolation effect of the c-2 continuous cubic Hermite interpolation function when the parameters were taken  $(\alpha, \beta) = (2, 2)$  was significantly better than that when the parameters were taken  $(\alpha, \beta) = (-2, -1)$  and  $(\alpha, \beta) = (-3, 1)$  . Therefore, it is necessary to discuss how to select appropriate  $\alpha$  and  $\beta$  values, so that the C~2 continuous cubic Hermite interpolation function  $s_i(x) (i = 0, 1, \dots, n-1)$  interpolated in the function  $y = f(x)$  has the best interpolation effect.

When parameter  $\alpha = \beta = 1$ , adjust and control the whole curve locally or as a whole, as shown in figure 2:

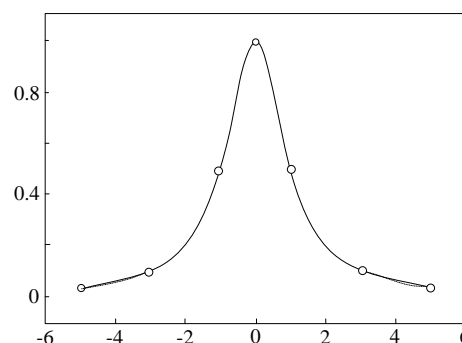


**Figure 2 Local or global regulation of the whole curve**

According to figure 2, when parameters  $\alpha$  and  $\beta$  take appropriate values, local or overall regulation of the whole curve can be realized. When the difference conditions were given, and under appropriate conditions, the cubic Hermite interpolation spline curves of C~2 were not only C~1, but also C~2.

For the above mentioned interpolation conditions,  $(\alpha, \beta) = (2.3466, 2.3706)$  can be obtained through calculation. In this case, the c-2 continuous cubic Hermite interpolation function obtained has the minimum global interpolation error. The curves of the optimal c-2 continuous cubic Hermite interpolation function and the interpolated function are shown in figure 3. The solid line is the curve of c-2 continuous cubic

Hermite interpolation function, and the dotted line is the curve of the interpolated function.



**FIG. 3 Optimal interpolation function and interpolated function**

Under the same interpolation conditions, the five-order Hermite interpolation spline function with second-order elliptic variation, the seven-order Hermite interpolation spline function with second-order elliptic variation and the three-order Hermite interpolation spline function with second-order elliptic variation were respectively used to interpolate function  $y = \frac{1}{1+x^2}, -5 \leq x \leq 5$ . The optimal values of parameters and the overall interpolation errors of different methods were shown in table 1.

**Table 1 Parameter optimal values and global interpolation errors of different methods**

Method	Parameter optimum	Interpolation error
Quintic Hermite interpolation spline function	(3.8787,5.0113)	$0.4551 \cdot 10^{-2}$
Seventh order Hermite interpolation spline function	(2.8271,1.3333)	$0.2974 \cdot 10^{-2}$
Cubic trigonometric Hermite interpolation spline function	(2.3466,2.7306)	$0.2032 \cdot 10^{-3}$

It can be seen from table 1 that for cubic Hermite interpolation spline

function  $y = \frac{1}{1+x^2}, -5 \leq x \leq 5$ , the c-2 continuous cubic Hermite interpolation

function with second-order elliptic variation proposed in this paper can be completely reconstructed when appropriate values of parameters  $\alpha$  and  $\beta$  are taken.

Example 2: suppose the interpolation function

$$y = f(x) = 4 - 2\sin x, 0 \leq x \leq 5\pi \tag{41}$$

Take

$$x_i = \frac{\pi}{2}i, y_i = f(x_i), m_i = f'(x_i) (i=0,1,\dots,10)$$

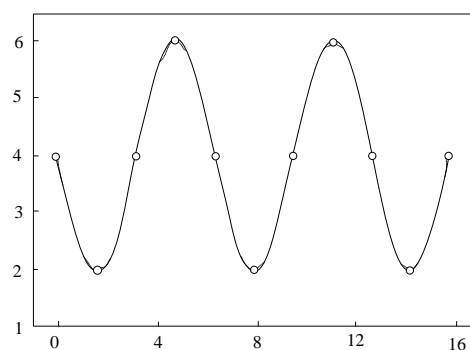
$(\alpha, \beta) = (-0.0196, -4.6248)$  can be obtained

by calculation, and the cubic Hermite interpolation spline function obtained at this time has the minimum global interpolation error. The curves of the optimal cubic Hermite interpolation spline function and the interpolated function drawn are shown in figure 4, where the solid line is the cubic Hermite interpolation spline function curve and the dotted line is the interpolated function curve.

**Table 2 Parameter optimal values and global interpolation errors of different methods**

Method	Parameter optimum	Interpolation error
Quintic Hermite interpolation spline function	(-0.0231, 0.0000)	$0.4079 \times 10^{-1}$
Seventh order Hermite interpolation spline function	(1.9939, 1.3333)	$0.6287 \times 10^{-1}$
Cubic trigonometric Hermite interpolation spline function	(-0.0196, -4.6248)	$0.2964 \times 10^{-1}$

It can be seen from table 2 that for cubic Hermite interpolation spline function  $y = f(x) = 4 - 2\sin x, 0 \leq x \leq 5\pi$ , when the parameters  $\alpha$  and  $\beta$  were



**FIG. 4 Optimal interpolation function and interpolated function**

Under the same interpolation conditions, the five-order Hermite interpolation spline function with second-order elliptic variation, the seven-order Hermite interpolation spline function with second-order elliptic variation, and the three-order Hermite interpolation spline function with second-order elliptic variation were respectively used to interpolate function  $y = f(x) = 4 - 2\sin x, 0 \leq x \leq 5\pi$ .

The optimal values of parameters and the overall interpolation errors of various methods were shown in table 2.

appropriate, the c-2 continuous cubic Hermite interpolation function with second-order elliptic variation proposed in this paper could be completely reconstructed.

Example 3: let the interpolated function

$$y = f(x) = x^7, 0 \leq x \leq 1 \tag{42}$$

Take

$$x_i = i, y_i = f(x_i), m_i = f'(x_i) (i = 0, 1)$$

Under the same interpolation conditions, the five-order Hermite interpolation spline function with second-order elliptic variation, the

seven-order Hermite interpolation spline function with second-order elliptic variation and the three-order Hermite interpolation spline function with second-order elliptic variation were respectively used to interpolate function  $y = f(x) = x^7, 0 \leq x \leq 1$ . The optimal parameter values and the overall interpolation errors of different methods were shown in table 3.

**Table 3 Parameter optimal values and global interpolation errors of different methods**

Method	Parameter optimum	Interpolation error
Quintic Hermite interpolation spline function	(0.0000, 1.0000)	$0.6800 \times 10^{-1}$
Seventh order Hermite interpolation spline function	(92.3514, -17.9122)	$0.4943 \times 10^{-3}$
Cubic trigonometric Hermite interpolation spline function	(21.0000, 5.0000)	0.0000

It can be seen from table 3 that for cubic Hermite interpolation spline function  $y = x^7, 0 \leq x \leq 1$ , the c-2 continuous cubic Hermite interpolation function with second-order elliptic variation proposed in this paper can be completely reconstructed when appropriate values of parameters  $\alpha$  and  $\beta$  are taken.

Example 4: determine the optimal values of parameters  $\alpha$  and  $\beta$ ,

according to the formula (36), respectively, using the second order elliptic changes of five Hermite interpolation spline function, seven times of changes of second order elliptic Hermite interpolation spline function and the change of second order elliptic triangle Hermite interpolation spline function to solve three times to get approximate optimal parameters and their energy  $E$ , as shown in table 4.

**Table 4 Optimal parameters and energy values of different methods**

Method	$p_i$	$p'_i$	$E$
Quintic Hermite interpolation spline function	-3.45	1.90	6.04
Seventh order Hermite	2.71	-0.41	5.82

interpolation spline function			
Cubic trigonometric			
Hermite interpolation spline function	-0.87	0.71	5.13

According to table 4 shows that the proposed changes of second order elliptic C ~ 2 continuous three triangle Hermite interpolation spline curve of energy value is lower than the second order elliptic changes of five Hermite interpolation spline curve, seven times of changes of second order elliptic Hermite interpolation spline curve of energy, according to the principle of fair, in this paper, the second order elliptic changes of C ~ 2 three consecutive triangle Hermite interpolation spline curve fairing is far better than that of second order elliptic changes five Hermite interpolation spline curve, seven times of changes of second order elliptic Hermite interpolation spline curve.

In conclusion, this paper puts forward the second order elliptic changes of C ~ 2 continuous three triangle Hermite interpolation function not only has good effect, the whole interpolation error is less than the second order elliptic changes of five Hermite interpolation spline function, seven times of changes of second order elliptic Hermite interpolation spline function, and the proposed changes of second order elliptic C ~ 2 continuous three trigonometric spline curve of Hermite interpolation optimal smoothing degree.

**4 Conclusion**

In order to solve the shortcomings of cubic Hermite interpolation spline in shape control and continuity, a c-2 continuous cubic Hermite interpolation spline with second-order elliptic

variation was designed. Not only can the spline automatically achieve C~2 continuity without any conditions, but also its shape can be regulated by the parameters, and the spline can meet the corresponding requirements according to the values determined by different targets. Furthermore, this paper also discusses the corresponding c-2 continuous cubic Hermite interpolation spline. In the case that c-2 continuous triangular Hermite interpolation spline is satisfied, the interpolation effect can be adjusted by the parameters. By determining the best values of the parameters, the interpolation function can have the best interpolation effect, and the smoothness of the c-2 continuous cubic Hermite interpolation spline curve can be improved.

**5. Data availability statement**

The data used to support the findings of this study are available from the corresponding author upon request.

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