Development of Vogel Method for Solving a Four Index Transportation Problem

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Abstract:
In this paper we deal with an extension of the Vogel method, which has been shown to be effective in finding an initial solution close to be optimal solution to the two index transportation problem for a four index problem, is denoted by Vogel4: In this work we intended also to compare the performance of Vogel4 with least cost method. The comparison has been found that the solution obtained by Vogel4 method is the closest to the optimal solution and that least cost method lost its effectiveness for a four index problem.

Keywords: transportation problem, multi-index problem, linear programming.

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1. Introduction:

Due the importance of transportation problem in real-world life, in industry, economy, telecommunication, localization and many others; there have been various studies on this subject. Since the first classical problem with two-dimensional index developed by Frank Lauren Hitchcock (1941) who also named, the transportation problem which transports goods from m sources to n different destinations to minimize total shifting cost. Important developments were made in this area during the second world war by mathematicians and economists: Koopmans in 1947, L.V. Kantorovich and M. K Gavourine in (1949 [6]) and then G.B Dantzig in (1951 [1]). Next the study has been extended to transportation problems with 3-index, 4-index and n-index [5], [7], [8], [9]: Determining an initial solution is an important step in solving the transportation problem. That why there are several methods proposed as least cost [1], [2] and Vogel [10].

In this paper, we are interested in the four index transportation problem, denoted by (TP4) with four indexes: origin, destination, merchandise types and vehicle types, one of extension cases of the Hitchcock is problem. The (TP4) includes the exchange of one or several merchandise types by vehicle types in a single link between origins to destinations. We will present the adapted algorithm from the Vogel method called vogel4 to obtain the initial basic feasible solution of the four index transportation problem. Vogel4 algorithm contains three steps: The first is for selecting the cell to allocate. In the second step the cell is selected for allocation and the selection criterion is the least expensive cell. The third step includes stop test. Section2 contains the four index transportation problem formulation and its economic interpretation. Section3 will be used to present the description of the new method Vogel4. In section 4 we find a comparative study. In section 5 we present results and conclusion.

2. Preliminaries:

2.1 Four index transportation problem formulation

The transportation problem with a four dimensional index, denoted by (TP4), is formulated as the following constrained optimization problem:

$$Minimize\ z = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\sum_{l=0}^{q} c_{ijkl}\, x_{ijkl}$$

Subject to the constraints:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \quad for\ all\ \ i = 1, \dots, m.$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \quad for\ all\ \ j = 1, \dots, n.$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \quad for\ all\ \ k = 1, \dots, p.$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \quad for\ all\ \ l = 1, \dots, q.$$

$$x_{ijkl} \geq 0 \qquad for\ all\ \ (i, j, k, l)$$

With: $\alpha_i > 0,\ \beta_j > 0,\ \gamma_k > 0,\ \delta_l > 0,\ c_{ijkl} \geq 0.$

This formulation is equivalent to the following linear program:

$$Minimize\ z = c^T x \quad s.c.\ Ax = b, x \geq 0.$$

With:

- $x = (x_{1111}, \dots, x_{mnpq})^{\mathrm{T}} \in \mathbb{R}^N.$
- $c = (c_{1111}, \dots, c_{mnpq})^{\mathrm{T}} \in \mathbb{R}^N.$
- $b = (\alpha_1, \dots, \alpha_m, ß_1, \dots, ß_n, \gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q)^{\mathrm{T}} \in \mathbb{R}^N.$
- A is a $M \times N$ matrix, where $M = m + n + p + q\ and\ N = mnpq.$

2.2 Definitions

- A feasible solution x of (TP4) is called basic solution if the columns of the sub-matrix A_x obtained from A by keeping only the columns corresponding to the variables $x_{ijkl} > 0$ are linearly independent.
- A basic feasible solution is said to be non-degenerate if

$$rank(A_x) = rank(A) = m + n + p + q - 3$$

2.3 Economic interpretation

Let:

$A_1, A_2, \dots, A_m$, m origins of availability $\alpha_1, \dots, \alpha_m$ (respectively).
$B_1, B_2, \dots, B_n$, n destinations of request $ß_1, \dots, ß_n$(respectively).
$S_1, S_2, \dots, S_P$, p vehicles types chosen appropriately of reserved load $\gamma_1, \dots, \gamma_p$ (respectively).
$H_1, H_2, \dots, H_q$, q goods types of quantities $\delta_1, \dots, \delta_q$(respectively).

3 Vogel4 method

In the following section we present the initialization methods: least cost4 and Vogel4 for solving the previous problem

3.1 Initialization procedure

It contains 3 steps:

Step 1: Choice of the cell to allocate.
Step 2: Assignment to the chosen cell.
Step 3: Stop test.

Unlike step1, steps 2 and 3 are the same for least cost4 and Vogel4 methods.

Step 2: If we chose the cell (i_0, j_0, k_0, l_0).

We take

$$x_{i_0 j_0 k_0 l_0} = \min(\alpha_{i_0}, ß_{j_0}, \gamma_{k_0}, \delta_{l_0})$$
$$\alpha_{i_0} = \alpha_{i_0} - x_{i_0 j_0 k_0 l_0}$$
$$ß_{j_0} = ß_{j_0} - x_{i_0 j_0 k_0 l_0}$$
$$\gamma_{k_0} = \gamma_{k_0} - x_{i_0 j_0 k_0 l_0}$$
$$\delta_{l_0} = \delta_{l_0} - x_{i_0 j_0 k_0 l_0}$$

For all (i, j, k, l) sush that x_{ijkl} is not determined:

If $\alpha_{i_0} = 0$ then $x_{i_0 jkl} = 0\ \forall (j, k, l) \neq (j_0, k_0, l_0)$
If $ß_{j_0} = 0$ then $x_{i j_0 kl} = 0\ \forall (i, k, l) \neq (i_0, k_0, l_0)$
If $\gamma_{k_0} = 0$ then $x_{ijk_0 l} = 0\ \forall (i, j, , l) \neq (i_0, j_0, , l_0)$
If $\delta_{l_0} = 0$ then $x_{ijkl_0} = 0\ \forall (i, j, k) \neq (i_0, j_0, k_0)$

Step 03: Repeat step 1 and 2 until all x_{ijkl} are determined.

Regarding step 1, we have:

3.1.1 Least cost4

Consists of choosing the least cost cell.

3.1.2 Vogel4

Let: $I = \{1, \dots, m\}$; $J = \{1, \dots, n\}$; $K = \{1, \dots, p\}$; $L = \{1, \dots, q\}$.

1/ Determine the penalties u_i, v_j, w_k, t_l for each: i∈ I; j∈ J; k∈ K; l∈ L as follows:

- for all $i \in I$ determine

$minc_{ijkl} = c_{ij_0k_0l_0}$; $minc_{ijkl} = c_{ij_1k_1l_1}, \forall (j,k,l) \neq (j_0,k_0,l_0)$;

$$u_i = c_{ij_1k_1l_1} - c_{ij_0k_0l_0}$$

- for all $j \in J$ determine:

$minc_{ijkl} = c_{i_0jk_0l_0}$; $minc_{ijkl} = c_{i_1jk_1l_1}, \forall (i,k,l) \neq (i_0,k_0,l_0)$;

$$u_i = c_{i_1jk_1l_1} - c_{i_0jk_0l_0}$$

- for all $k \in K$ determine:

$minc_{ijkl} = c_{i_0j_0kl_0}$; $minc_{ijkl} = c_{i_1j_1kl_1}, \forall (i,j,l) \neq (i_0,j_0,l_0)$;

$$u_i = c_{i_1j_1kl_1} - c_{i_0j_0kl_0}$$

- for all $l \in L$ determine:

$minc_{ijkl} = c_{i_0j_0k_0l}$; $minc_{ijkl} = c_{i_1j_1k_1l}, \forall (i,j,k) \neq (i_0,j_0,k_0)$;

$$u_i = c_{i_1j_1k_1l} - c_{i_0j_0k_0l}$$

2/ Identify among the u_i, v_j, w_k, t_l the dimension corresponding to the highest penalty.

3/ Determine in the chosen dimension of penalties the cell (i,j,k,l) with the smallest cost.

4/ If there is a tie in penalties; choose from among them the dimension with the cell the smallest cost.

5/ If there is again a tie (least cost cells), chose the first dimension on the left.

Example:

$$m = n = p = q = 2$$

$\alpha_1 = 8$	$ß_1 = 8$	$\gamma_1 = 7$	$\delta_1 = 7$
$\alpha_2 = 2$	$ß_2 = 2$	$\gamma_2 = 3$	$\delta_2 = 3$

$ijkl$	1111	1112	1121	1122	1211	1212	1221	1222
c_{ijkl}	14	28	40	21	71	24	41	12
$ijkl$	2111	2112	2121	2122	2211	2212	2221	2222
c_{ijkl}	76	24	67	84	42 1	77	71	1

Iteration1:

Step1:

1.1 Determine penalties:

$$\begin{cases} i = 1: \min_1 c_{1jkl} = c_{1222} = 12, & \min_2 c_{1jkl} = c_{1111} = 14, & u_1 = 2 \\ i = 2: \min_1 c_{2jkl} = c_{2222} = 1, & \min_2 c_{2jkl} = c_{2112} = 24, & u_2 = 7 \end{cases}$$

$$\begin{cases} j = 1: \min_1 c_{i1kl} = c_{1111} = 14, & \min_2 c_{i1kl} = c_{1122} = 21, & v_1 = 7 \\ j = 2: \min_1 c_{i2kl} = c_{2222} = 1, & \min_2 c_{i2kl} = c_{1222} = 12, & v_2 = 11 \end{cases}$$

$$\begin{cases} k = 1: \min_1 c_{ij1l} = c_{1111} = 14, & \min_2 c_{ij1l} = c_{1212} = 24, & w_1 = 10 \\ k = 2: \min_1 c_{ij2l} = c_{2222} = 1, & \min_2 c_{ij2l} = c_{1222} = 12, & w_2 = 11 \end{cases}$$

$$\begin{cases} l=1:\ \min\limits_{1} c_{ijk1} = c_{1111} = 14, & \min\limits_{2} c_{ijk1} = c_{1121} = 40, & t_1 = 26 \\ l=2:\ \min\limits_{1} c_{ijk2} = c_{2222} = 1, & \min\limits_{2} c_{ijk2} = c_{1222} = 12, & t_2 = 11 \end{cases}$$

1.2. The greatest penalty is $t_1 = 26$ so $l = 1$

1.3. The cell with the smallest cost for $l = 1$ is (1111).

Step2: Allocation to the chosen cell

$$x_{1111} = \min(\alpha_1, ß_1, \gamma_1, \delta_1) = \min(8,8,7,7) = 7$$

$$\alpha_1 = \alpha_1 - x_{1111}$$
$$ß_1 = ß_1 - x_{1111}$$
$$\gamma_1 = \gamma_1 - x_{1111}$$
$$\delta_1 = \delta_1 - x_{1111}$$

$\alpha_1 = 1$	$ß_1 = 1$	$\gamma_1 = 0$	$\delta_1 = 0$
$\alpha_2 = 2$	$ß_2 = 2$	$\gamma_2 = 3$	$\delta_2 = 3$

$$x_{ij1l} = 0 \ \ and \ \ x_{ijk1} = 0$$

Iteration2:

Step1:

1.1 Determine penalties:

$$\begin{cases} i=1:\ \min\limits_{1} c_{1jkl} = 12, & \min\limits_{2} c_{1jkl} = 21, & u_1 = 9 \\ i=2:\ \min\limits_{1} c_{2jkl} = 1, & \min\limits_{2} c_{2jkl} = 84, & u_2 = 83 \end{cases}$$

$$\begin{cases} j=1:\ \min\limits_{1} c_{i1kl} = 21, & \min\limits_{2} c_{i1kl} = 84, & v_1 = 63 \\ j=2:\ \min\limits_{1} c_{i2kl} = 1, & \min\limits_{2} c_{i2kl} = 12, & v_2 = 11 \end{cases}$$

$$\begin{cases} k=1:\ \ / \\ k=2:\ \min\limits_{1} c_{ij2l} = 1, & \min\limits_{2} c_{ij2l} = 12, & w_2 = 11 \end{cases}$$

$$\begin{cases} l=1:\ \ / \\ l=2:\ \min\limits_{1} c_{ijk2} = 1, & \min\limits_{2} c_{ijk2} = 12, & t_2 = 11 \end{cases}$$

1.2. The greatest penalty is $u_2 = 83$ so $i = 2$

1.3. The cell with the smallest cost for $i = 2$ is (2222).

Step2: Allocation to the chosen cell

$$x_{2222} = \min(\alpha_2, ß_2, \gamma_2, \delta_2) = \min(2,2,3,3) = 2$$
$$\alpha_2 = \alpha_2 - x_{2222}$$
$$ß_2 = ß_2 - x_{2222}$$
$$\gamma_2 = \gamma_2 - x_{2222}$$
$$\delta_2 = \delta_2 - x_{2222}$$

$\alpha_1 = 1$	$ß_1 = 1$	$\gamma_1 = 0$	$\delta_1 = 0$
$\alpha_2 = 0$	$ß_2 = 0$	$\gamma_2 = 1$	$\delta_2 = 1$

$$x_{2jkl} = 0 \ \ and \ \ x_{i2kl} = 0$$

Iteration3:

Only cell (1122) remains, so $x_{1122} = 1$

$\alpha_1 = 0$	$ß_1 = 0$	$\gamma_1 = 0$	$\delta_1 = 0$
$\alpha_2 = 0$	$ß_2 = 0$	$\gamma_2 = 0$	$\delta_2 = 0$

Step3: stop.

4 Comparative study:

We take:

T_1: the average phase1 execution time for 10 instances.

T_2: the average phase 2 execution time for 10 instances.

T_{total}: the average execution time of the two phases for 10 instances.

$Val_{initial}$: the average of the initial phase1 value for 10 instances.

$Val_{optimal}$: the average of the optimal phase2 value for 10 instances.

Nbr: the average number of phase 2 iteration for 10 instances.

- For the comparison we used algorithm ALPT4 [13] in phase 2.

Example 1:

Size: 8x6

	T_1	T_2	T_{total}
Least cost4	00:00	00:00	00:00
Vogel 4	00:0001	00:00	00:0001

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	242909,1	261390,8	1,4
Vogel 4	232927,3	261390,8	1 ,3

	Percentage of prbs degenerating
Least cost4	00
Vogel 4	00

Example 2:

Size: 22x900

	T_1	T_2	T_{total}
Least cost4	00 : 0349	00 : 0083	00 : 0432
Vogel 4	00 : 1612	00 : 0079	00 : 1691

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	1310189,9	496485,3504	26,9
Vogel 4	1375539,3	496485,3504	25,3

	Percentage of prbs degenerating
Least cost4	00
Vogel 4	30

Example 3:

Size: 34x5184

	T_1	T_2	T_{total}
Least cost4	01 : 536	00 : 1899	01 : 7259
Vogel 4	06 : 6483	00 : 1493	06 : 7976

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	26391631,500	4943838,3181	72,3
Vogel 4	20880857,000	4943838,3181	64,0

	Percentage of prbs degenerating
Least cost4	20
Vogel 4	10

Example 4:

Size: 38x8100

	T_1	T_2	T_{total}
Least cost4	4 : 5963	00 : 3581	04 : 9544
Vogel 4	17 : 2181	00 : 3585	17 : 5766

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	23872695,4	3332000,9981	87,5
Vogel 4	26244949,3	3332000,9981	88,1

	Percentage of prbs degenerating
Least cost4	00
Vogel 4	00

Example 5:

Size: 40 x 9900

	T_1	T_2	T_{total}
Least cost4	5 : 296	00 : 4551	5 : 7511
Vogel 4	30 : 5668	00 : 5817	31 : 14855

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	1838,875	35036,75	89,25
Vogel 4	1929,375	35036,75	84,125

	Percentage of prbs degenerating
Least cost4	70
Vogel 4	80

Example 6:

Size: 40 x 10000

	T_1	T_2	T_{total}
Least cost4	5 : 951	0 : 5443	6 : 4953
Vogel 4	34 : 5752	00 : 5117	35 : 0929

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	3139,7	584,5904	104,5
Vogel 4	2548;1	584,5904	98,3

	Percentage of prbs degenerating
Least cost4	80
Vogel 4	90

Example 7:

Size: 43 x 13068

	T_1	T_2	T_{total}
Least cost4	11 : 0738	00 : 7631	11 : 8369
Vogel 4	47 : 8196	00 : 8235	48 : 6431

	$Val_{initial}$	$Val_{optimal}$	Nbr
Least cost4	224617,7	21959,1418	121,5
Vogel 4	196611,3	21959,1418	112,6

	Percentage of prbs degenerating
Least cost4	30
Vogel 4	00

5 Results and conclusion

We used algorithm ALPT4 for the comparison

- In T_1: least cost4 is better than vogel4.
- In T_2: vogel4 is better than least cost4.
- In T_{total}: We find that the time saved by vogel4 in T_2 does not compensate for the loss in T1which makes it ranked last.
- In Nbr: Vogel4 is best followed by least cost4.
- By contribution least cost4: it remains the best method; percentage of degenerating problems is 20%.

References

[1] G.B.DANTZIG, linear programming and extensions, linear programming and extensions 1963.

[2] G.B.DANTZIG, application of the simplex method to a transportation problem, Activity analysis of production and allocation; Koopmans, T.C.Ed., John Wiley and Sons, New York, 1951.

[3] S.G Diagne, Y.Gningue, Méthode de Vogel Modifier pour la résolution du problème de transport simple. Applied Mathematical sciences, Vol.5, 2011, no.48, 2373-2388

[4] Tuyet-Hoa Pham and Philippe Dott, An exact method for solving the four index transportation problem and industrial application, Opr. Res 3 (2), (2013), 28-44.

[5] K.B. Hally, The multi-index transportation problem, Opr. Res., 11 (1963), 368-79.

[6] M.K. Kantarovitch, M.K. Gavurin, Application of mathematical methods to problems of analysis of freight flows, Problems of raising the efficiency of transport performance, Moscow-Leningrad, (in Russian), 1949.

[7] M.K. Kravtsov, A counter example to the hypothesis of maximum number of integer vertices of multi-index axial transportation polyhedron, Discrete Math.App., 10(2000) no.1, pp. 109-114.

[8] M.K. Kravtsov, E. V. Lukshin, On the non-integer polyhedron vertices of three index axial transportation problem, Autom. Remote control, 65(2004) no. 3, PP. 422-430.

[9] M. Queyranne, F. C. R. Spieksma, Approximation algorithms for multi-index transportation problems with decomposable costs, Discrete Appl.Math., 76 (1997), pp. 239-253.

[10] N.V. Reinfeld and W.R.Vogel, Mathematical programming. Prentice-Hall, Englewood cliffs, N.J. (1958).

[11] Tuyet-Hoa Pham and Philippe Dott, An exact method for solving the four index transportation problem and industrial application, American journal of operational research, vol. 3 (2), (2013), 28-44.

[12] R. Zitouni and A. Keraghel, Resolution of the capacitated transportation problem with four subscripts, Kybernetes, 32,9/10, pp.1450-1463 (2003).

[13] R. Zitouni, Etude qualitative de modèle de transport et localisation, Thèse de Doctorat de mathématiques appliquées, Université Ferhat Abbas, Sétif, (2007).

[14] R. Zitouni and M. Achache, A numerical comparison between two exact simplicial methods for solving a capacitated 4-index transportation problem, J. Numer. Anal. Approx. Theory, vol.46 no. 2, pp.181-192 (2017).